Goal: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Idea: Recognize $f(t) \Longleftrightarrow F(s)$ pairs

Put $F(s)$ in standard form

$F(s), I(s), V(s)$ is "rational function" (ratio of polynomials)

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \dots b_1 s + b_0}$$

First: if $n \geq m$ (improper rational function)

$$F(s) = \frac{s^2+1}{s} = s + \frac{1}{s}$$

$$F(s) = \frac{s^3 + 3s^2 + 4s + 7}{s+1}$$

$$= s^2 + \frac{\cancel{s^3} + 3s^2 + 4s + 7 - s^2(\cancel{s}+1)}{s+1}$$

$$= s^2 + \frac{2s^2 + 4s + 7}{s+1}$$

$$= s^2 + 2s + \frac{\cancel{2s^2} + 4s + 7 - 2s(\cancel{s}+1)}{s+1}$$

$$= s^2 + 2s + \frac{2s + 7}{s+1}$$

$$= s^2 + 2s + 2 + \left.\underline{\frac{s}{s+1}}\right\}\text{proper}$$

$$f(t) = \delta''(t) + 2\delta'(t) + 2\delta(t) + 5e^{-t}u(t)$$

Partial Fraction Expansion

$$s(s+2)^2(s+1) = s^4 + 5s^3 + 8s^2 + 4s$$

Example $F(s) = \dfrac{s^2+5s+2}{s^4+5s^3+8s^2+4s}$

$$\frac{(s^2)(s^3+5s^2+8s+4)}{s^2(s+2)} = \frac{\cancel{3s^2}}{s^2} + \frac{(3s^2+8s+4)3s}{3s(s+2)}$$

$$= s^2 + 3s + \frac{2s+4}{s+2} = s^2 + 3s + 2$$

$$= (s+1)(s+2)$$

$$F(s) = \frac{s^2 + 5s + 2}{s(s+2)^2(s+1)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{(s+2)^2} + \frac{K_4}{(s+2)}$$

Denominator m^{th}-order

Numerator $m-1$ - order (or <u>less</u>)

m coefficients (some may be zero)

$$\left.\begin{array}{l} m \text{ equations} \\ m \text{ unknowns} \end{array}\right\} \text{solve}$$

$$f(t) = \left(K_1 + K_2 e^{-t} + K_3 t e^{-2t} + K_4 e^{-2t}\right) u(t)$$

Step 1: Make proper

Step 2: Factor denominator

Step 3: ~~For~~ Partial fraction expansion: Find Ks

(a) <u>Distinct real roots</u> $\left(\frac{K_i}{s+a_i}\right)$

$$\frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{(s+2)^2} + \frac{K_4}{s+2}$$

K_2: multiply by $s+a$, set $s=-a$

$$(s+1)\left[\frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{(s+2)^2} + \frac{K_4}{s+2}\right]$$

$$\left[\cancel{\frac{K_1(s+1)}{s}}^{\,0} + \underline{K_2} + \cancel{\frac{K_3(s+1)}{(s+2)^2}}^{\,0} + \cancel{\frac{K_4(s+1)}{(s+2)}}^{\,0}\right]_{\underline{s=-1}}$$

$$\cancel{(s+1)}\left[\frac{s^2+5s+2}{s\cancel{(s+1)}(s+2)^2}\right]_{s=-1} = \frac{1-5+2}{(-1)(1)^2} = \underline{2}$$

(b) Repeated real roots

Multiply by $(s+a)^n$ set $s=-a$

$$\left\{(s+2)^2\left[\frac{K_1}{s}+\frac{K_2}{(s+1)}+\frac{K_3}{(s+2)^2}+\frac{K_4}{s+2}\right]\right\}_{s=-2} = K_3$$

$$\left.\frac{(s^2+5s+2)\cancel{(s+2)^2}}{(s^2+s)\cancel{(s+2)^2}}\right|_{s=-2} = K_3$$

$$\frac{4-10+2}{4-2} = \frac{-4}{2} = -2 = K_3$$

$$K_4 = \left[\frac{d}{ds}\left[\frac{s^2+5s+2}{s^2+s}\right]\right]_{s=-2}$$

$F(s)\,(s+2)^2$

$$K_4 = \left[\frac{(s^2+s)(2s+5)-(2s+1)(s^2+5s+2)}{(s^2+s)^2}\right]_{s=-2}$$

$$= \frac{(2)(1)-(-3)(-4)}{4} = \frac{-10}{4} = -\frac{5}{2}$$

$$\frac{s^2+5s+2}{s^4+5s^3+8s^2+4s} = \frac{1/2}{s} + \frac{2}{s+1} + \frac{(-2)}{(s+2)^2} + \frac{(-\frac{5}{2})}{s+2}$$

<u>Check your answer!</u> $s=1$

$$\frac{1+5+2}{1+5+8+4} = \frac{8}{18} \stackrel{?}{=} \frac{1}{2} + \frac{2}{2} - \frac{2}{9} - \frac{5}{6}$$

$$= \frac{9+18-4-15}{18} = \frac{8}{18} \checkmark$$

$$f(t) = \frac{1}{2}u(t) + 2e^{-t}u(t) - 2te^{-2t}u(t) - \frac{5}{2}e^{-2t}u(t)$$

$$= u(t)\left[\frac{1}{2} + 2e^{-t} - 2te^{-2t} - \frac{5}{2}e^{-2t}\right]$$

$$\frac{bs^2+cs+d}{(s+a)^3} = \frac{K_1}{(s+a)^3} + \frac{K_2}{(s+a)^2} + \frac{K_3}{(s+a)}$$

$$\frac{d^2}{ds^2}\left(\frac{K_3 (s+a)^{\cancel{3}\,2}}{\cancel{(s+a)}}\right) = \frac{d}{ds}\left[K_3\, 2(s+a)\right]$$

$$= 2K_3 \; [$$

$$\frac{d^3}{ds^2}(s+a)^3$$

$$\frac{d}{ds^2}\left[3 \overset{K_4}{(s+a)^2}\right] = \frac{d}{ds}\left\{\left[3 \times 2 (s+a)\right] K_4\right\}$$

$$= 6K_4$$

Distinct Complex Roots

$$F(s) = \frac{10}{s^2 + 4s + 5}$$

$$\frac{-4 \pm \sqrt{(4)^2 - 4(5)}}{2}$$

$$= -2 \pm \sqrt{-1}$$

$$= -2 \pm j = -\alpha \pm j\beta$$

$$= \frac{10}{(s+2+j)(s+2-j)}$$

$$= \frac{K_1}{s+2+j} + \frac{K_2}{s+2-j} = \frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$$

$$\mathcal{L}^{-1}\{F(s)\} = 2|K|e^{-\alpha t}\cos(\beta t + \theta) \qquad K = |K|e^{j\theta}$$

$$K = \left(\frac{10\,\cancel{(s+2-j)}}{(s+2+j)\cancel{(s+2-j)}} \right) \Bigg|_{s=-2+j}$$

$$= \frac{10}{\cancel{-2}+\cancel{2}j+\cancel{2}+j} = \frac{10}{j2} = \frac{5}{j} = -j5$$

$$5\angle -90^\circ$$

$$f(t) = 2 * 5\, e^{-2t} \cos(t - 90^\circ)$$

$$\underline{10\, e^{-2t} \cos(t - 90^\circ)}$$

$s = 0$

$$\frac{10}{(s^2+4s+5)} = \frac{-j5}{(s+2-j)} + \frac{j5}{(s+2+j)} = 2 \checkmark$$

$$2 = \frac{-j5}{2-j} + \frac{j5}{2+j} = \frac{(2+j)(-j5)+(2-j)j5}{5}$$

Repeated complex roots $\Rightarrow$ Just like real

$$\frac{d}{ds}\left[F(s)(s+\alpha-j\beta)^2\right]_{s=-\alpha+j\beta}$$

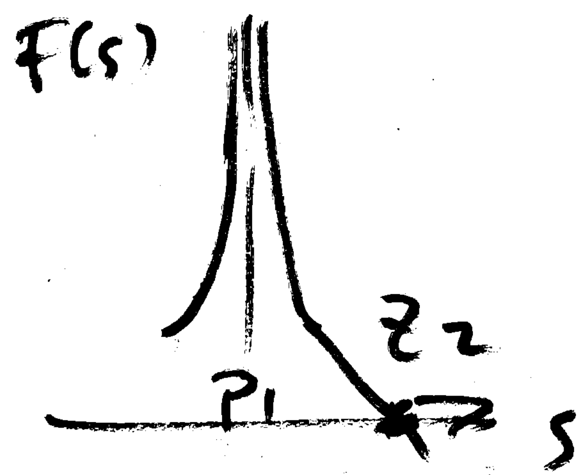
$F(s)$		$f(t)$
$\frac{K}{s+a}$	$\Longleftrightarrow$	$Ke^{-at}u(t)$
$\frac{K}{(s+a)^2}$	$\Longleftrightarrow$	$Kte^{-at}u(t)$
$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$\Longleftrightarrow$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$
$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$\Longleftrightarrow$	$2\lvert K\rvert te^{-\alpha t}\cos(\beta t+\theta)u(t)$

Poles and zeroes

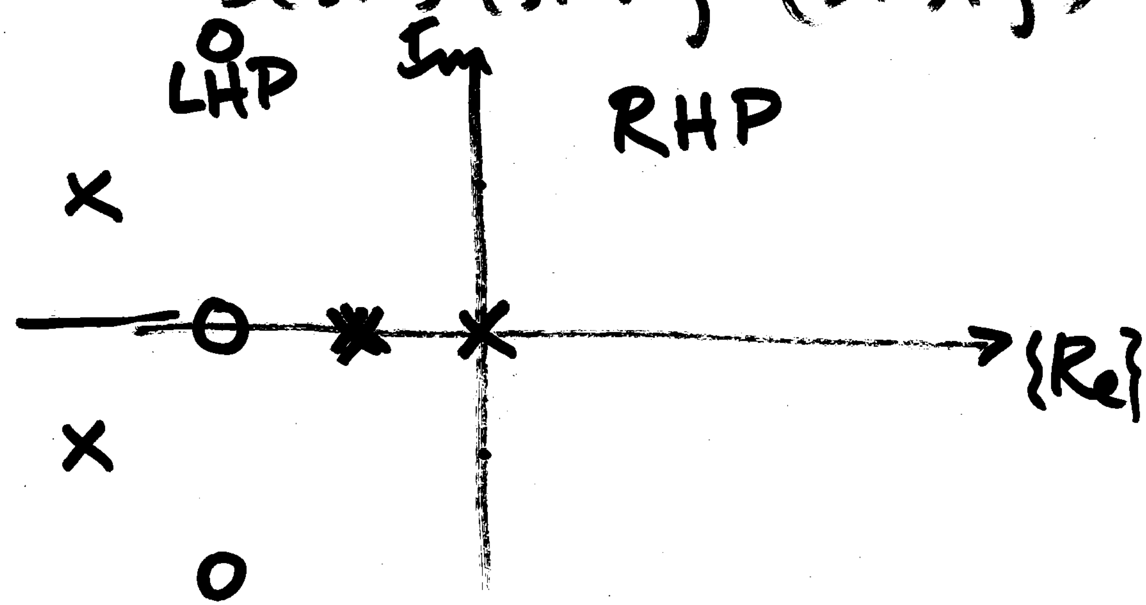
$$F(s) = K \frac{(s \mp z_1)(s \mp z_2) \cdots (s \mp z_n)}{(s \mp p_1)(s \mp p_2) \cdots (s \mp p_m)}$$

real or complex

roots of denominator



$$F(s) = \frac{K(s+2)(s+2-j2)(s+2+j2)}{S(s+1)^2(s+3-j)(s+3+j)}$$



Initial & final value theorem's

IVT: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$ [if $f(t)$ doesn't contain impulse function]

FVT: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ [All poles in LHP, except up to 1 @ origin]

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt$$

$$\lim_{s\to\infty} \left[sF(s) - f(0^-)\right] = \lim_{s\to\infty} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt \qquad \overset{\text{no impulse}}{t > 0} \quad 0$$

$$= \lim_{s\to\infty} \left[\int_{0^-}^{0^+} \frac{df}{dt} e^{-st}\, dt + \cancel{\int_{0^+}^{\infty} \frac{df}{dt} e^{-st}\, dt}\right]$$

$$\underline{\lim_{s\to\infty} \left[sF(s)\right]} - \cancel{f(0^-)} = f(0^+) - \cancel{f(0^-)} \qquad \text{no impulse @ origin}$$

$$= \underline{\lim_{t\to 0^+} f(t)}$$

$$\lim_{s \to 0} \left[sF(s) - f(0^-) \right] = \lim_{s \to 0} \int_{0^-}^{\infty} \frac{df}{dt} e^{\cancel{-st}}_{\,1} \, dt$$

$$= \cancel{\lim} \int_{0^-}^{\infty} \frac{df}{dt} \, dt$$

$$\lim_{s \to 0} \left[sF(s) \right] \cancel{- f(0^-)} = \lim_{t \to \infty} f(t) \cancel{- f(0^-)}$$

Final value theorem

Example

$$F(s) = \frac{7s^2 + 63s + 134}{(s+3)(s+4)(s+5)}$$

$$f(t) = \left(4e^{-3t} + 6e^{-4t} - 3e^{-5t}\right) u(t)$$

IVT: $\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \left(\frac{7s^3 + 63s^2 + 134s}{s^3 + \_ s^2 + \_ s + \_} \right) = 7$

$$\lim_{t \to 0^+} f(t) = 4 + 6 - 3 = 7 \checkmark$$

FVT: $\lim_{s \to 0} \left[s F(s) \right] = \lim_{t \to \infty} f(t) = 0$

$$\lim_{s \to 0} \left\{ \frac{s(7s^2 + 63s + 134)}{(s+3)(s+4)(s+5)} \right\} = \frac{0\,(134)}{3 \cdot 4 \cdot 5} = 0$$

$$\mathcal{L}\left\{ \frac{\partial}{\partial t} \cdots \int dt \cdots \right\} \Rightarrow F(s) = 0$$

General method for Linear D-I Equations

For us focus on circuits

sources, R, C, L

$$V = \mathcal{L}\{v\} \; \& \; I = \mathcal{L}\{i(t)\}$$

$$V(s) \longleftrightarrow v(t) \qquad I(s) \longleftrightarrow i(t)$$

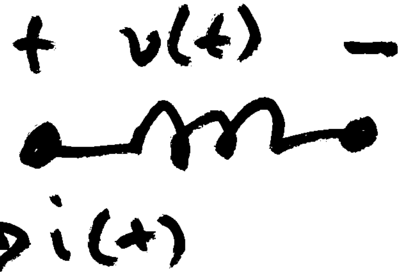
$$\mathcal{L}\{v = iR\} \Rightarrow V(s) = R\,I(s)$$

Inductor $\mathcal{L}\left\{ v = L\frac{di}{dt} \right\}$

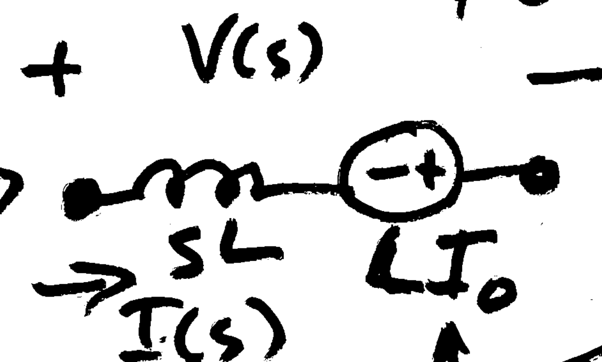
$V(s) = L\left[sI(s) - i(0^-) \right]$

$V(s) = \underbrace{sL}_{\text{like } Z} I(s) - LI_0 \qquad i(0^-) = I_0$

$\uparrow \neq 0$ if stored energy

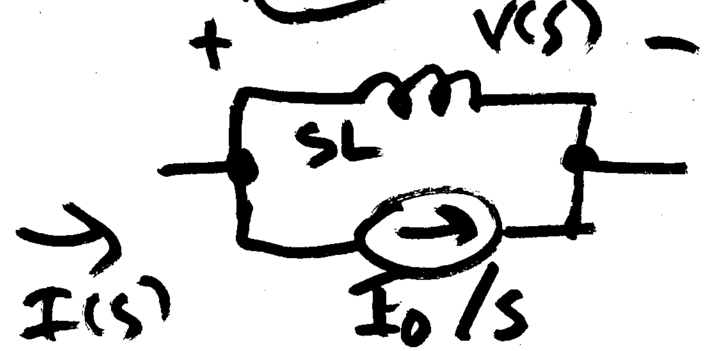


$\Rightarrow$



note sign

$\frac{V(s) + LI_0}{sL} = I(s)$

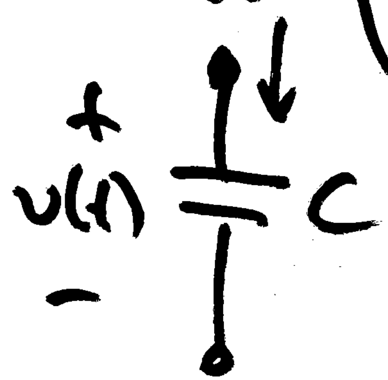


Capacitor $\mathcal{L}\left\{ i = C\frac{dv}{dt} \right\}$

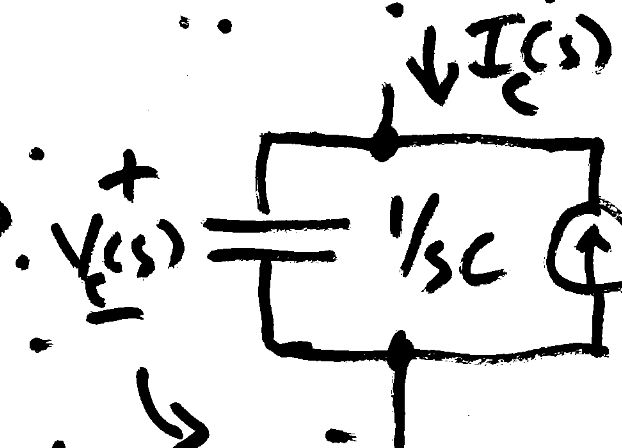
$V_0 = v(0^-)$

$$I(s) = C\left[sV(s) - v(0^-)\right]$$

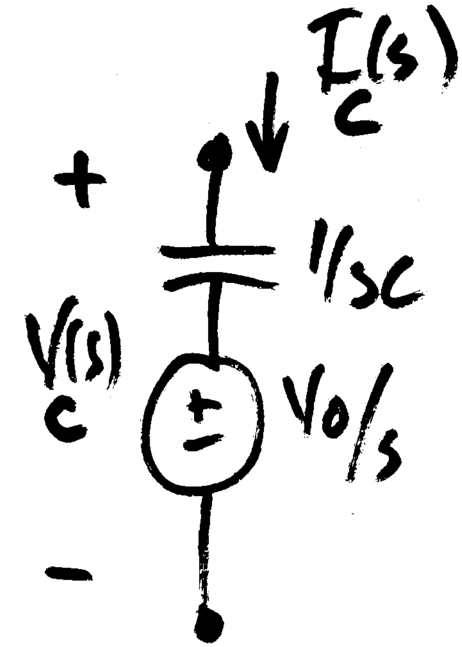
$$= sCV(s) - CV_0$$



$\Rightarrow$



or



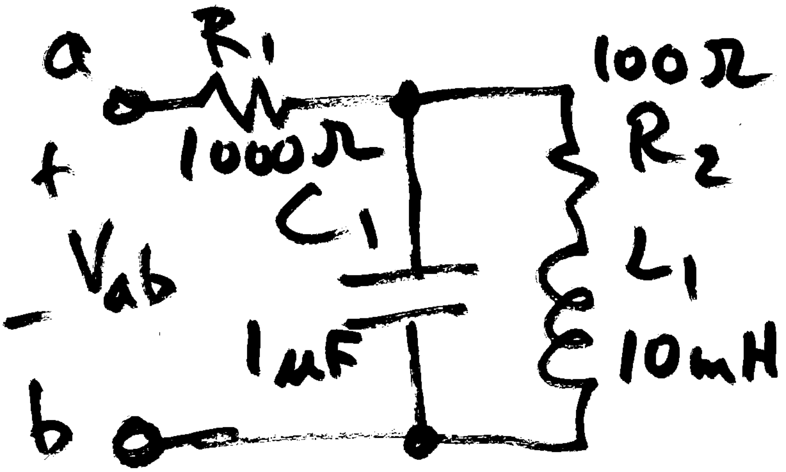
$V(s) / 1/sC$
$= sC\,V(s)$

$$V(s) = \frac{I_s + CV_0}{sC} = \frac{I_s}{sC} + \frac{V_0}{s}$$

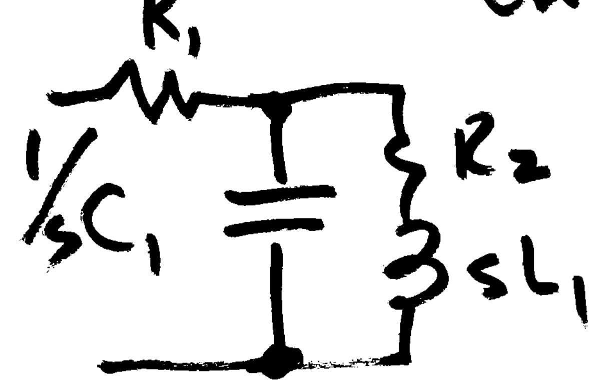
Transform entire circuit, use KCL, KVL

<u>Example</u>: (a) Find Z_{ab}, no initial energy in devices

$I_0, V_0 = 0$



$\Rightarrow$



$$Z_2 = \frac{1}{sC_1} \parallel (R_2 + sL_1) = \frac{\left[\frac{1}{sC_1}(R_2 + sL_1)\right] sC_1}{\left[R_2 + sL_1 + 1/sC_1\right] sC_1}$$

$$= \frac{sL_1 + R_2}{s^2 L_1 C_1 + sR_2 C_1 + 1}$$

$$Z_{ab} = R_1 + Z_2$$

$$Z_{ab} = \frac{s^2 L_1 C_1 R_1 + s R_1 R_2 C_1 + \underline{R_1 + R_2} + s L_1}{s^2 L_1 C_1 + s R_2 C_1 + 1}$$

$$= \frac{10^{-5} s^2 + (0.11) s + 1100}{10^{-8} s^2 + 10^{-4} s + 1}$$

$$Z_{ab} = \frac{1000\left(s^2 + 1.1\times 10^4 s + 1.1\times 10^8\right)}{s^2 + 10^4 s + 10^8}$$

(b) Find poles & zeroes of Z_{ab}

Poles roots of $s^2 + 10^4 s + 10^8 \Rightarrow s = \dfrac{-10^4 \pm \sqrt{10^8 - 4 \cdot 10^8}}{2}$

poles $\dfrac{10^4}{2}\left(-1 \pm j\sqrt{3}\right)$ $\quad = -5\times 10^3 \pm \dfrac{10^4\sqrt{3}\, j}{2}$

Zeroes @re roots of $s^2 + 1.1 \times 10^4 s + 1.1 \times 10^8$

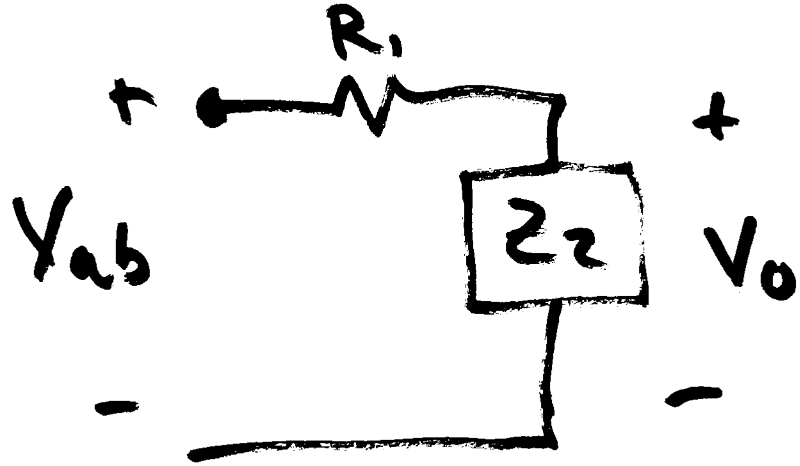
$$s = \frac{-1.1 \times 10^4 \pm \sqrt{(1.1)^2 - 4.4}\ 10^4}{2}$$

$$= \frac{10^4}{2}\left[-1.1 \pm j\sqrt{\cancel{0.0121 - 0.}\ \underbrace{1.21 - 4.4}_{1.986}}\right]$$

$$= 10^4\left[-0.55 \pm j0.893\right]$$

2 complex conjugate poles

2 " " zeroes



Transfer function

$$\frac{V_o}{V_{ab}} = \frac{Z_2}{R_1 + Z_2} = \frac{Z_2}{Z_{ab}} = H(s)$$

$$H(s) = \frac{V_o}{V_{ab}} = \frac{sL_1 + R_2}{s^2 L_1 C_1 R_1 + s R_1 R_2 C_1 + sL_1 + R_1 + R_2}$$

poles $= 10^4(-0.55 \pm j\,0.893)$

zero @ $s = -R_2/L_1 = -10^4$

$$2|K|\, e^{-5.5\times10^3 t} \cos(8.93\times10^3 t + \phi)$$