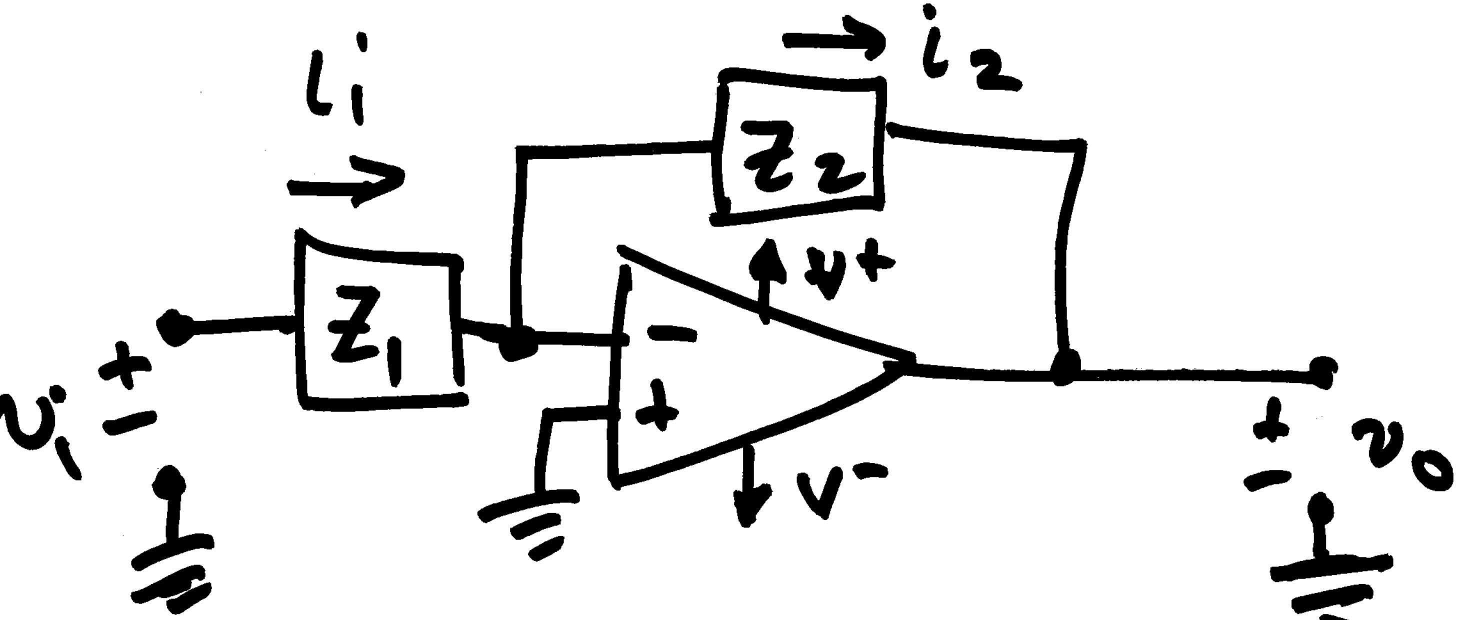
R, L, C $\Rightarrow$ Passive Filter

Gain ≤ 1

Subject to Loading

2nd order High Q requires both L & C



(1) Assume op-amp is in linear regime

a) $V_+ - V_- = 0$

b) $i_+, i_- = 0$

c) Very high gain

(2). Analyze circuit

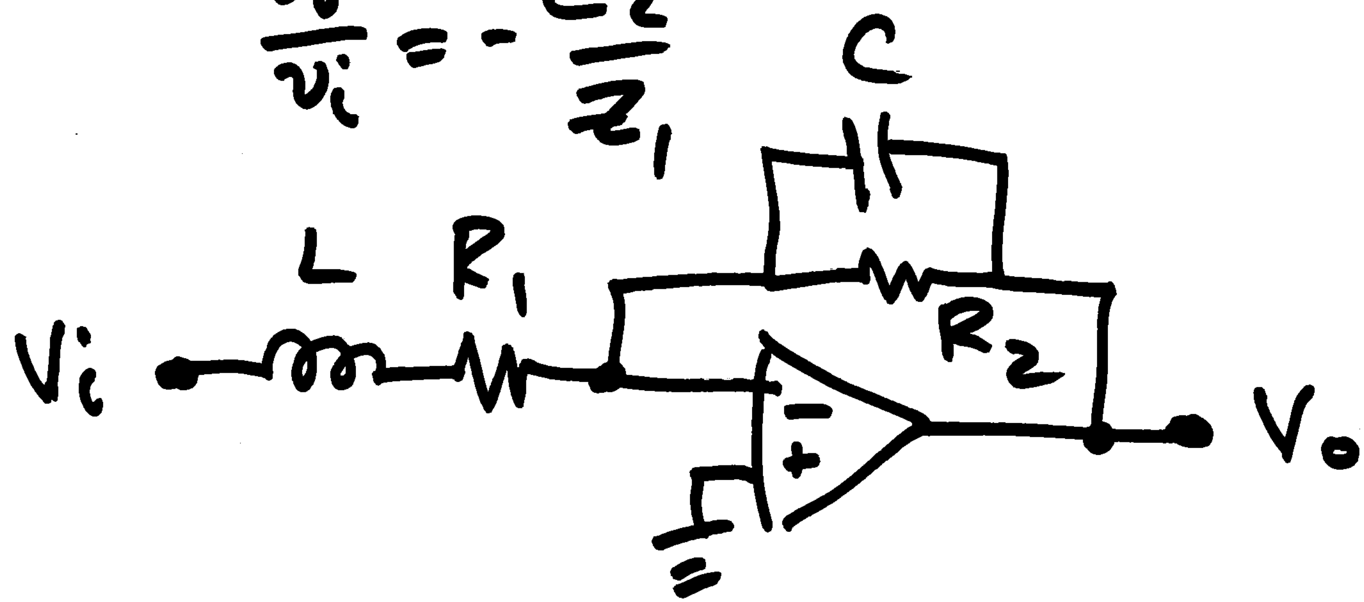
(3.) Check (1) valid (if not, assume at limit & repeat)

$$V_+ = 0 \Rightarrow V_- = 0$$

$$i_1 = \frac{v_i}{Z_1} = i_2$$

$$V_o = -Z_2 i_2 = -\frac{Z_2}{Z_1} v_i$$

$$\frac{v_o}{v_i} = -\frac{Z_2}{Z_1}$$



$Z_1 = R_1 + sL$

$Z_2 = \dfrac{1}{sC + 1/R_2} = \dfrac{1/C}{s + 1/R_2C}$

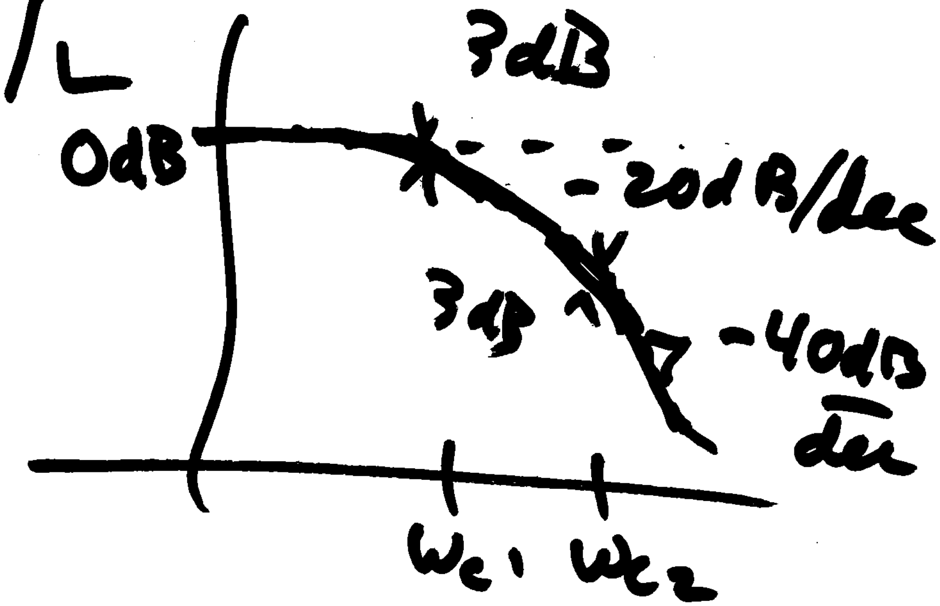
$$\frac{V_o}{V_i} = -\frac{1/LC}{(s + 1/(R_2C))(s + R_1/L)}$$

2nd order LPF

2 real roots : $-1/(R_2C)$, $-R_1/L$

$\omega_c = 1/R_2C$, R_1/L

ω_{c1}, ω_{c2}

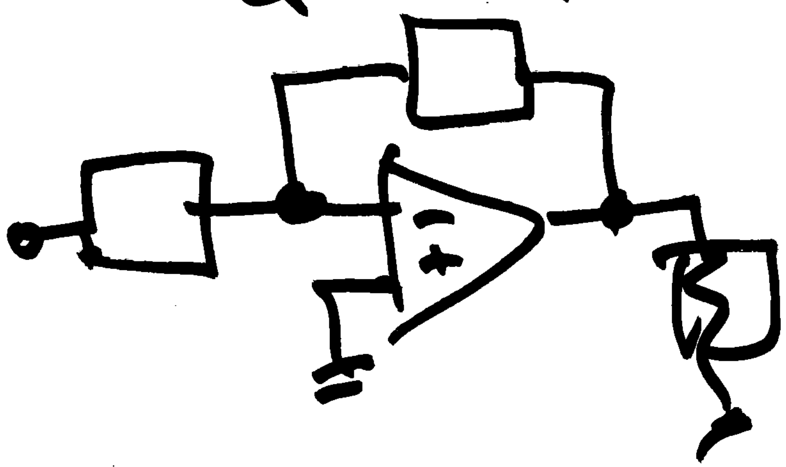


$R_1 = R_2$

$$\frac{V_o}{V_i} = \frac{-1/LC}{s^2 + \underbrace{\left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)}_{2\zeta\omega_0 = 2\alpha = b = \frac{\omega_0}{Q}} s + \underbrace{\frac{R_1}{R_2 LC}}_{\omega_0^2}}$$

$b = BW \geq 2\omega_0$

$Q \leq 1/2$



$Z_L \Rightarrow$ no change in V_o

Scaling:

Example $R_1 = 1\,\Omega = R_2$

$C = 1F,\ L = 1H$

Magnitude scaling $k_m * Z \Rightarrow$ ~~circuit~~ filter unchanged

$\frac{V_o}{V_i}$ or $\frac{I_o}{I_i}$ $\quad Z \times \frac{1}{Z} \Rightarrow \underline{I} \times 2$

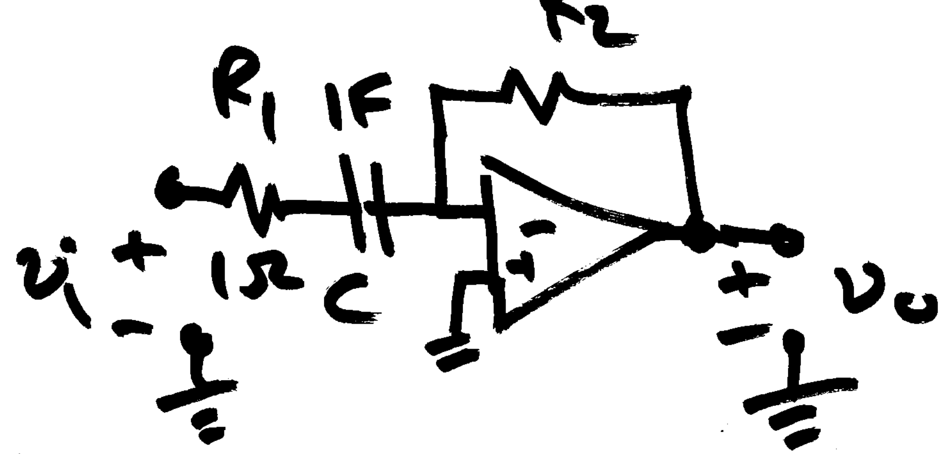
$R' = k_m R \quad L' = k_m L \quad C' = C/k_m$

$Z_c = 1/sC$

Frequency: k_f $\quad Z_R = R,\ Z_L = sL,\ Z_C = \frac{1}{sC}$

$R' = R,\ L' = L/k_f \ \&\ C' = C/k_f$

$R' = k_m R$, $L' = \frac{k_m}{k_f} L$, $C' = C/k_m k_f$



$$\frac{V_o}{V_i} = -\frac{Z_2}{Z_1} = -\frac{R_2}{1+1/s} = \frac{-sR_2}{s+1}$$

HPF corner/cutoff

10^6 rad/s $\Rightarrow k_f = 10^6$

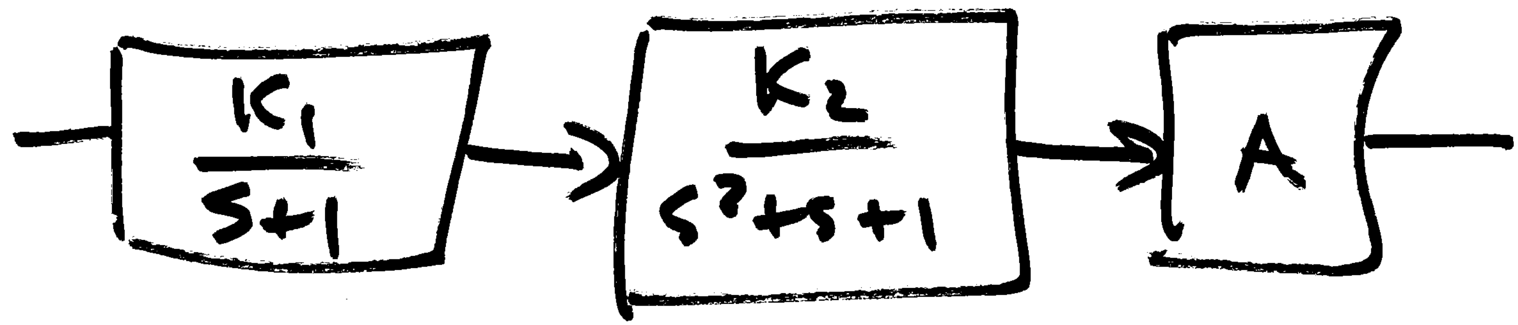
R Gain 26 dB = (20 + 6) dB $\Rightarrow 10 \times 2 = 20$

$R_1 = 1\Omega$, $R_2 = 20\ \Omega$, $C' = 1F/10^6 = 1\mu F$

$R_1: 1\Omega \Rightarrow 1k\Omega$ $k_m = 10^3$

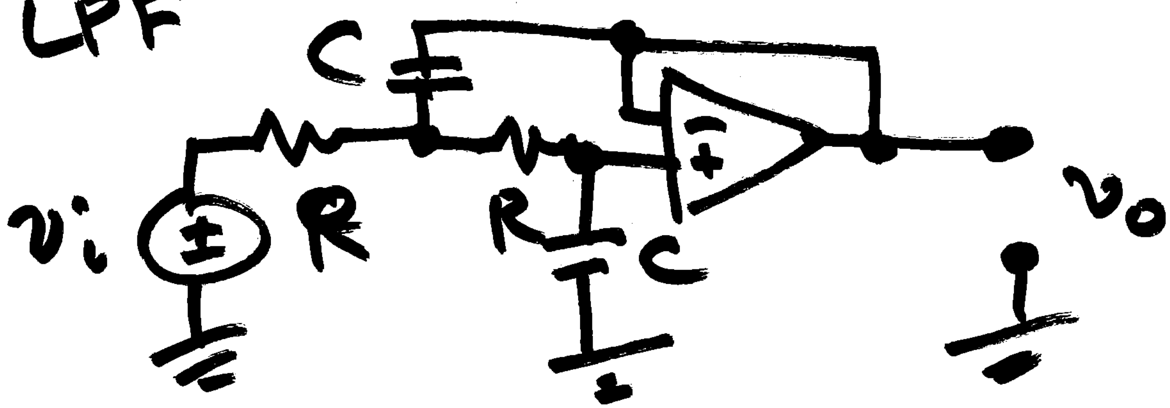
$R_2' = 20k\Omega$, $C'' = \frac{1\mu F}{10^3} = 1nF$

Higher Order Filters

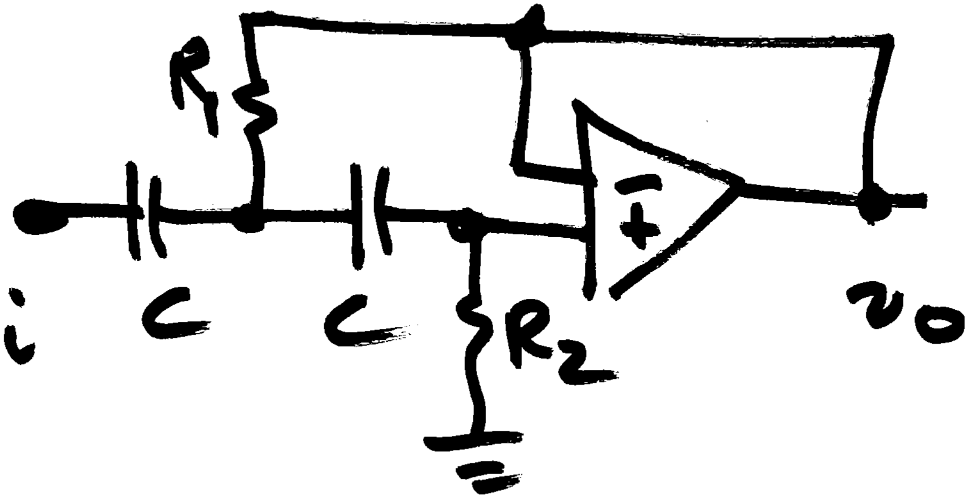


3rd order LPF

LPF



HPF



$$H(s) = \frac{s^2}{s^2 + \left(\frac{2s}{R_2 C}\right) + \left(\frac{1}{R_1 R_2 C^2}\right)}$$

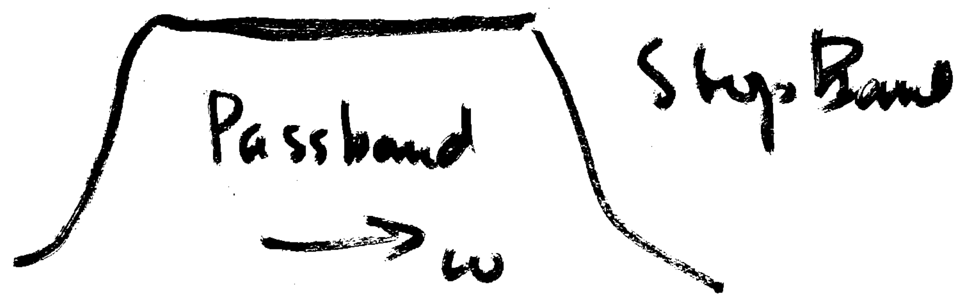
$$\omega_0 = \frac{1}{C}\frac{1}{\sqrt{R_1 R_2}}$$

$$\frac{1}{\cancel{C}\, Q \sqrt{R_1 R_2}} = \frac{\omega_0}{Q} = \frac{2}{R_2 \cancel{C}} \Rightarrow Q = \sqrt{\frac{R_2}{R_1}}\left(\frac{1}{2}\right)$$

$$\omega_0 = 1 : C = 1F, \quad \frac{1}{R_1 R_1} = 1\,\Omega^{-2} \quad b = 2/R_2$$

Designing a Filter

Wide band



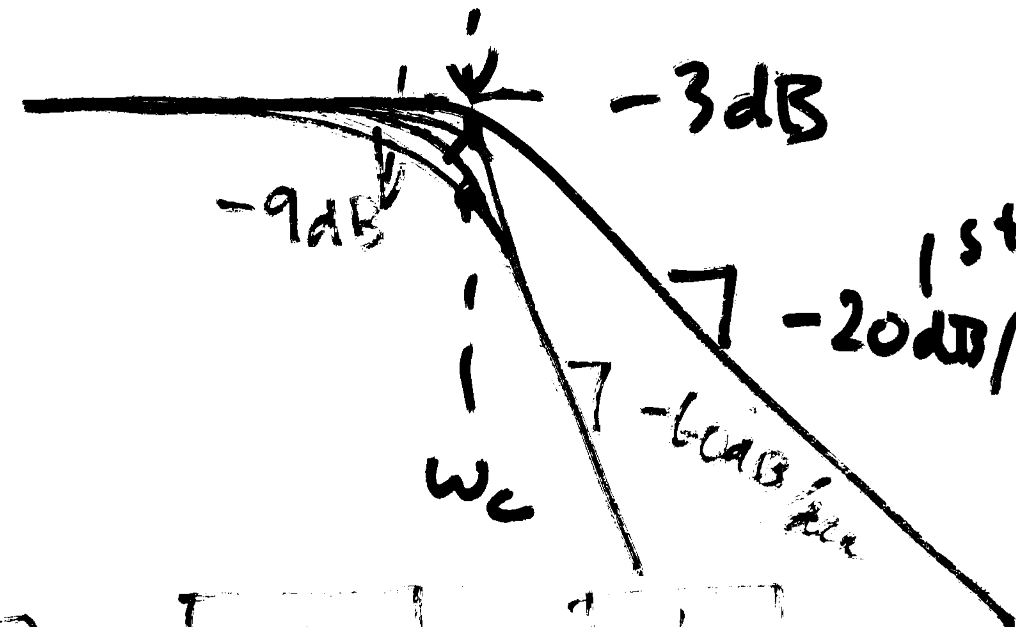
LPF ⇒ HPL ⇒ BPF

Narrowband: Pick center frequency $\frac{1}{\sqrt{LC}}$

Then $\frac{1}{RC}$ or $\frac{R}{L}$ to give desired Q

$$\left(s^2 + \frac{\omega_0}{Q} + \omega_0^2\right)$$

LPF

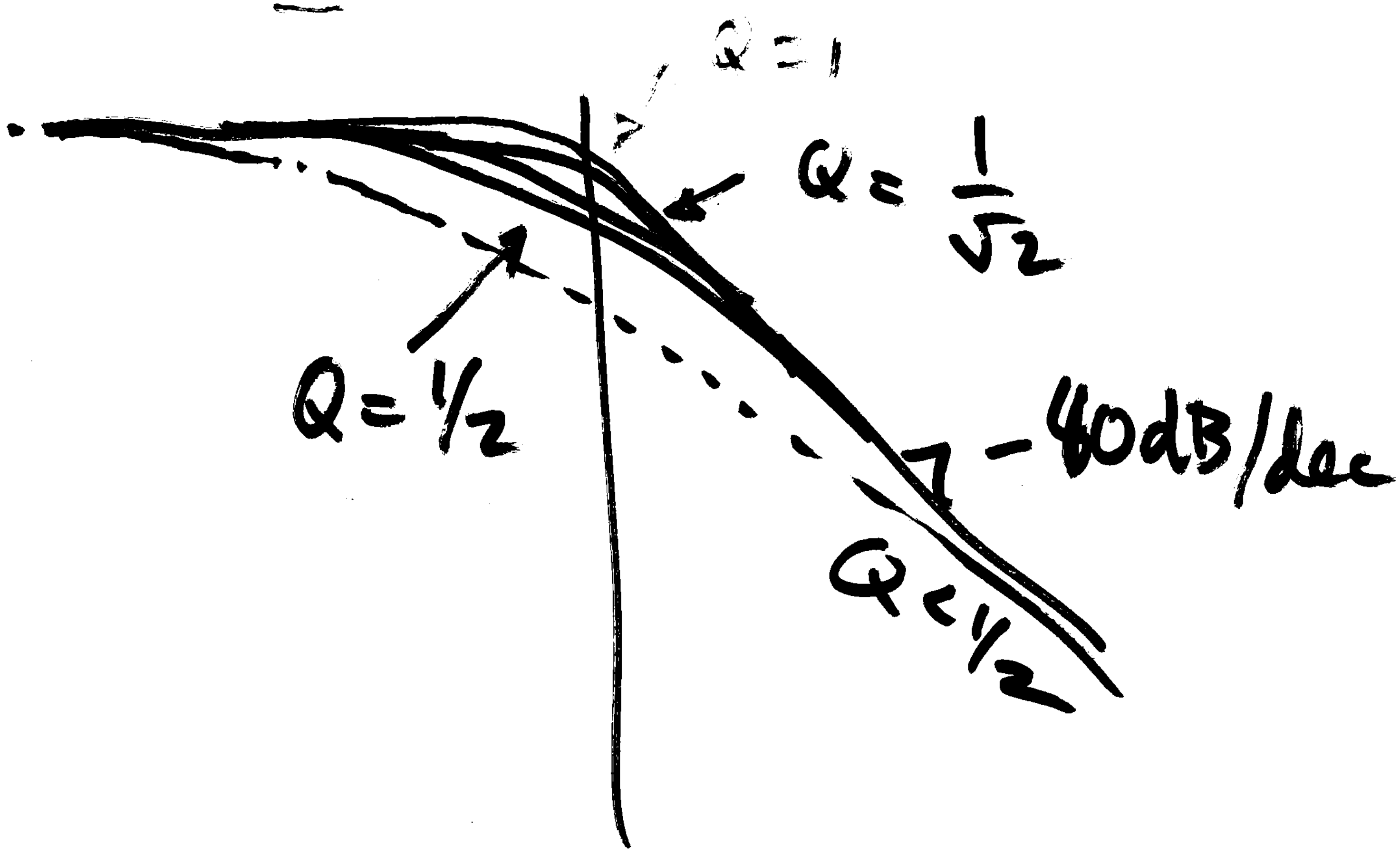


$$\frac{\alpha}{s+\alpha}$$

$$\boxed{\frac{1}{s+1}} \to \boxed{\frac{1}{s+1}} \to \boxed{\frac{1}{s+1}} \to \qquad \frac{1}{(s+1)^3}$$

2nd order LPF

$$\frac{\omega_0^2}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}$$



Butterworth Filters

Multiple 1st order filter $H(s) = \frac{-\omega_n^n}{(s+\omega_n)^n}$

$$H(s) = \frac{(-1)^n \omega_n^n}{(s+\omega_n)^n}$$

$$|H(j\omega)| = \left| \frac{\omega_n^n}{(\omega_n + j\omega)^n} \right| = \frac{1}{\sqrt{1+(\omega/\omega_n)^2}^{\,n}}$$

Butterworth $|H(j\omega)| = \frac{1}{\sqrt{\varepsilon\, 1+(\omega/\omega_c)^{2n}}}$

1st order $\dfrac{1}{s+1}$

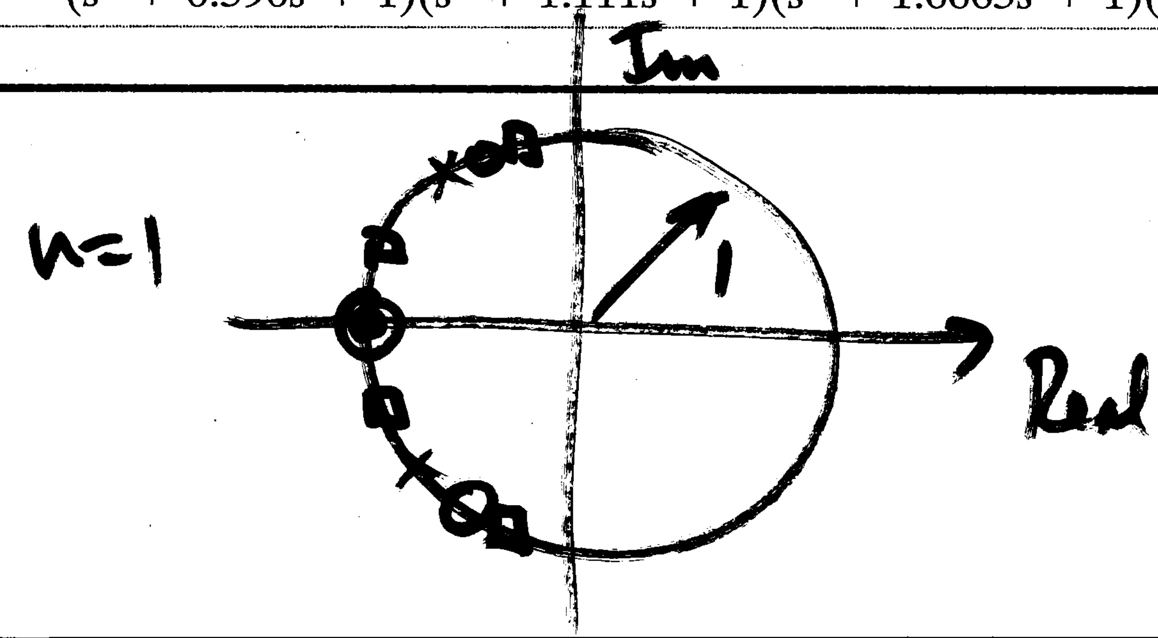
2nd order $\dfrac{1}{s^2+\sqrt{2}s+1}$ $Q=\dfrac{1}{\sqrt{2}}$

3rd order $\dfrac{1}{(s+1)(s^2+s+1)}$

$\underset{-3\,dB}{(s+1)}$ $\underset{Q=1 \Rightarrow 0\,dB\ drop}{(s^2+s+1)}$

TABLE 15.1 Normalized (so that $\omega_c = 1$ rad/s) Butterworth Polynomials up to the Eighth Order

n	nth-Order Butterworth Polynomial
1	$(s + 1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + \sqrt{2} + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.6663s + 1)(s^2 + 1.962s + 1)$



$$|H(j\omega)|^2 = H(j\omega)H(-j\omega)| = \frac{1}{1+\left[\left(\frac{\omega}{\omega_c}\right)^2\right]^n}$$

$$s = j\omega \Rightarrow \omega^2 = -s^2$$

$$|H(j\omega)|^2 = \frac{1}{1+\left[\left(\frac{s}{\omega_c}\right)^2\right]^n(-1)^n} = H(s)H(-s)$$

$$\omega_c = 1$$

$$\frac{1}{1+(-1)^n s^{2n}} = H(s)H(-s)$$

$$H(s) = \frac{1}{D(s)}$$

Butterworth HPF

$$\frac{s^n}{\cancel{s}\, D_n(s)}$$

<u>LPF</u>

$$\frac{1}{D_n(s)}$$

How high a filter order?

$$|H(j\omega)| = \frac{A}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

$$A_{dB}(\omega) - A_{dB}(\text{passband}) = 20\log_{10}\frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

$\omega >> \omega_c$ $$\cong -20n\log_{10}\left(\frac{\omega}{\omega_c}\right)$$

$A_{dB}(\omega_1) < A_s$ $$n > \frac{-A_s}{20\log_{10}\left(\frac{\omega_1}{\omega_c}\right)}$$

$\omega_c = 10^5\ \text{rad/s}$

* $A_{dB}(PB) = 0\text{dB}$

$A_{dB}(10^6\ \text{rad/s}) = -30\text{dB}$

$$n > \frac{-(-30)}{20(1)}$$

> 1.5

$$A = -10 \log_{10}\left[1 + \left(\frac{\omega}{\omega_c}\right)^{2n}\right]$$

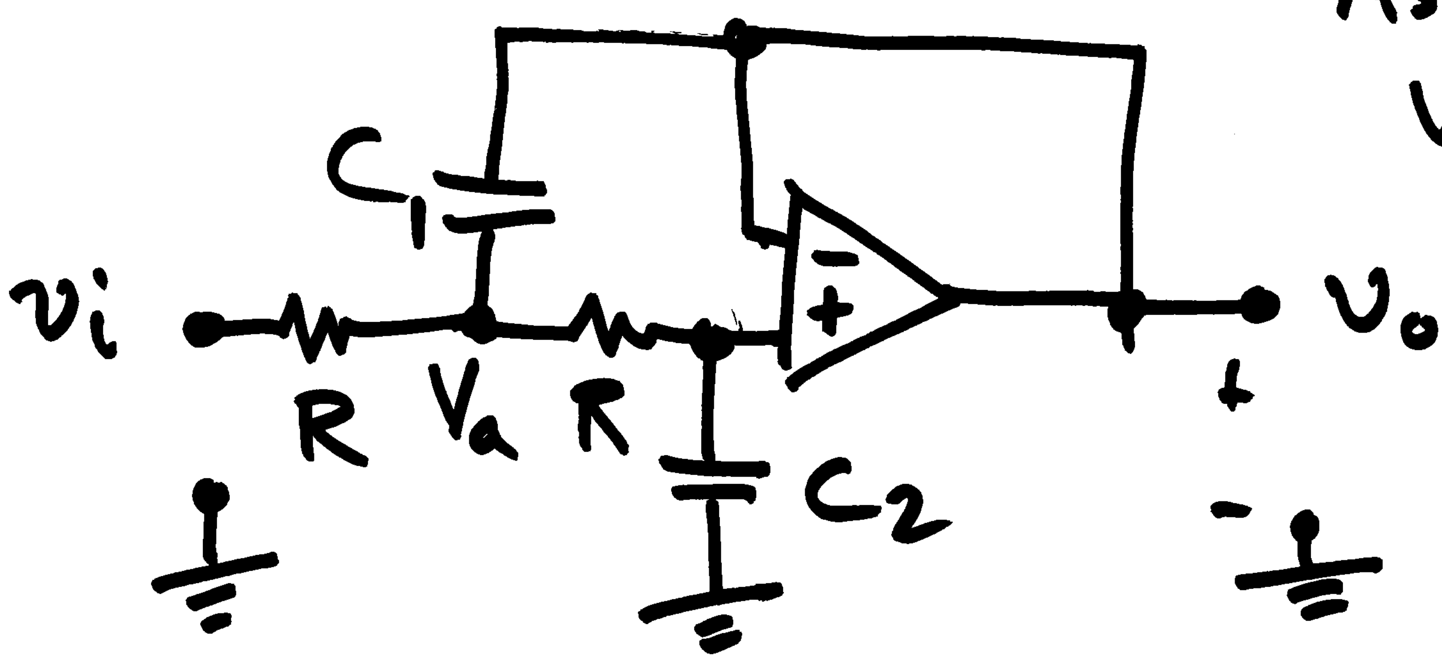
$$10^{-A/10} = 1 + \left(\frac{\omega}{\omega_c}\right)^{2n}$$

$$\left(\frac{\omega}{\omega_c}\right)^{2n} = 10^{-A/10} - 1$$

$$2n \log_{10}\left(\frac{\omega}{\omega_c}\right) = \log_{10}\left(10^{-A/10} - 1\right)$$

ΔA_{min} dB

$$n \geq \frac{\log_{10}\left(10^{-A/10} - 1\right)}{2\log_{10}(\omega/\omega_c)}$$



Assume linear

$V_+ = V_-$

$i_+ = i_- = 0$

$V_- = V_o = V_+$

$$V_a: \quad \frac{\underline{V_i - V_a}}{R} + \frac{\underline{V_o - V_a}}{1/sC_1} + \frac{V_o - V_a}{R} = 0$$

$$\underline{V_o} = \frac{1/sC_2}{R + 1/sC_2} V_a$$

$$\frac{V_o}{V_i} = \frac{1/R^2C_1C_2}{s^2 + \underbrace{\frac{2}{RC_1}}_{\frac{\omega_o}{Q}} s + \frac{1}{R^2C_1C_2}} \qquad \omega_o^2$$

LPF
2nd order

$\omega_o = 1$
$R = 1\,\Omega$

$$H(s) = \frac{1/C_1C_2}{s^2 + \frac{2}{C_1}s + \frac{1}{C_1C_2}}$$

$$\frac{1}{C_1C_2} = 1 = \omega_o^2$$

$$\frac{2}{C_1} = \frac{\omega_o}{Q} = \frac{1}{Q}$$

Butterworth $Q = \frac{1}{\sqrt{2}}$ $\quad \frac{2}{C_1} = \sqrt{2} \Rightarrow C_1 = \sqrt{2}$

$C_2 = 1/\sqrt{2}$

$\omega_0 = 10^5$ $\quad R = 1\,\Omega$

$k_f = 10^5$ $\quad C_1 = \sqrt{2}/10^5 = 14.14\,\mu F$

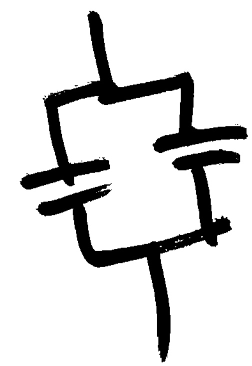
$C_2 = 1/\sqrt{2} \times 10^5 = 7.07\,\mu F$

$k_m = 10^3$ $\quad R = 1\,k\Omega$

$C_1 = 14.14\,nF$

$C_2 = 7.07\,nF$

$C_{kit} = C_2$ $\qquad C_1 = 2C_{kit}$



Filter Design

1. Determine specs (e.g. LPF, BPF gains PB or SB
2. Choose $H(s) \Rightarrow H(j\omega)$ to meet specs
3. Decompose 1st & 2nd order terms/stages
4. Choose circuits for each stage and cascade (or sum for band reject)
5. Correct gain, scale component (magnitude, frequency)
6. Check result

LPF w/ corner @ 10^4 rad/s $\qquad 20\log_{10} 50$

Pass band gain of 50 $\Rightarrow$ 40dB - 6dB = 34dB

<u>Gain at corner freq. of 37dB</u>

Gain less than 0.5 at ~~0.8x10~~ 8×10^4 rad/s

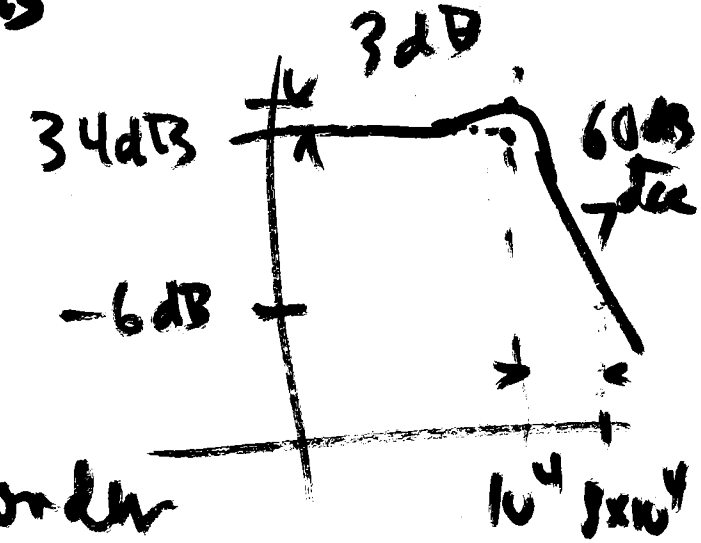
$$20\log_{10}(0.5) = -6\text{dB}$$

$\log_{10} 8 = 0.9$ dec (0.9 bel)

$34-(-6) = 40$dB drop

over 0.9 decades

$\frac{40}{0.9} = 44.4$ dB/decade $\Rightarrow$ 3rd order



60 dB/dec

$$H(s) = \frac{A}{(s+10^4)\left[s^2+\frac{10^4}{Q}s+(10^4)^2\right]}$$

$Q = \sqrt{2}$

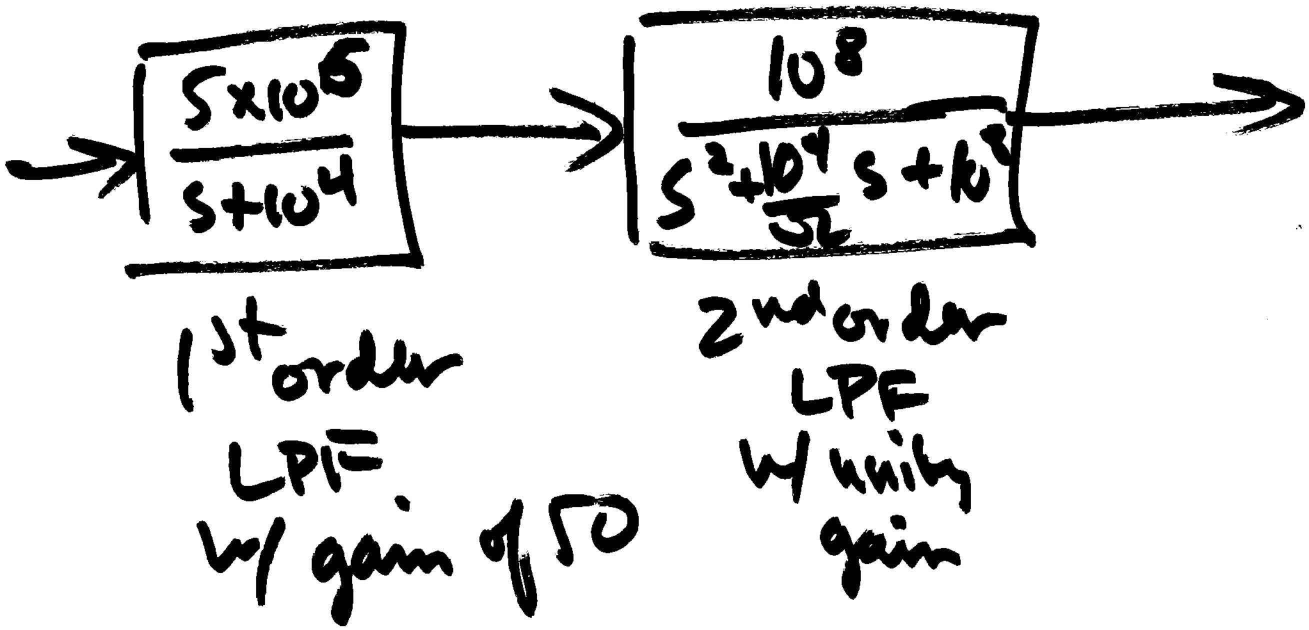
$$\lim_{\omega \to 0} |H(j\omega)| = \frac{A}{10^4 \times 10^8} = \frac{A}{10^{12}} = 50 \Rightarrow A = 5\times 10^{13}$$

$$H(s) = \frac{5\times 10^{13}}{(s+10^4)\left[s^2 + \frac{10^4}{\sqrt{2}}s + 10^8\right]}$$

Check $$|H(j\,8\times 10^4)| = \frac{5\times 10^{13}/10^{12}}{(j8+1)\left(-8^2 + \frac{j8}{\sqrt{2}} + 1\right)}$$

$$= \left.\frac{50}{(1+j8)(-63+j4\sqrt{2})}\right\} < 0.5$$

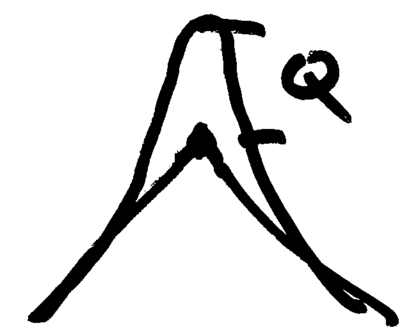
$$|\,| > 8 \times |\,| > 63 \quad > \quad > 100$$


$\frac{5\times10^5}{s+10^4}$
$\frac{10^8}{s^2+\frac{10^4}{\sqrt{2}}s+10^8}$
1st order LPF w/ gain of 50
2nd order LPF w/ unity gain

$$H(s) = \frac{A}{(s+10^4)^3}$$

Gain @ 10^4 rad/s
25 dB

$$H(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$



$$H(j\omega) = \frac{\omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q}j\omega + \omega_0^2}$$

$$H(j\omega_0) = \frac{\omega_0^2}{\cancel{-\omega_0^2} + j\frac{\omega_0^2}{Q} + \cancel{\omega_0^2}} = -jQ$$

$$|H(j\omega_0)| = Q$$

$$\angle H(j\omega_0) = -90^\circ$$

LPF

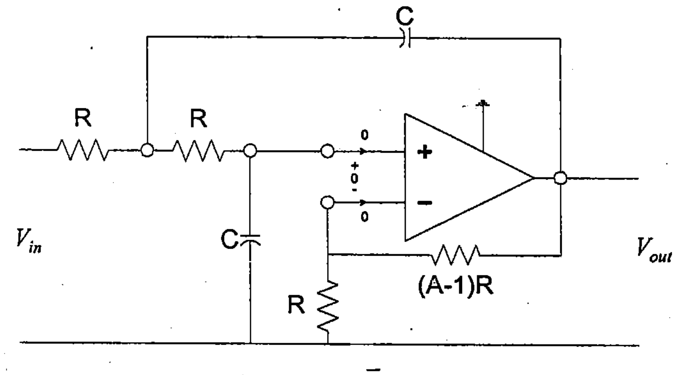
$$\frac{A_v \omega_c^2}{s^2 + \frac{\omega_c}{Q} s + \omega_c^2}$$

s^2 , $\frac{\omega_c}{Q} s$

HPF BPF

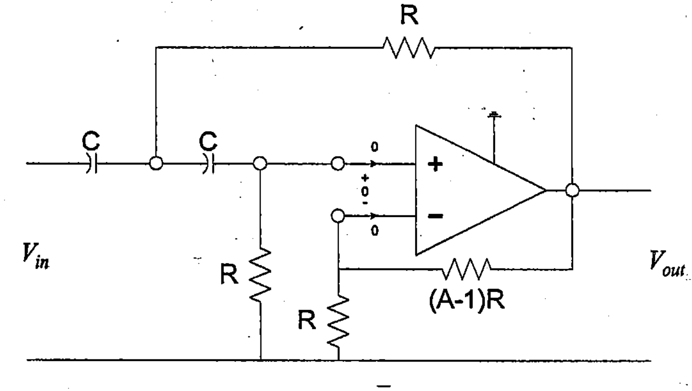
Second-Order Sallen-Key Filters

S.K. Low-Pass Filter



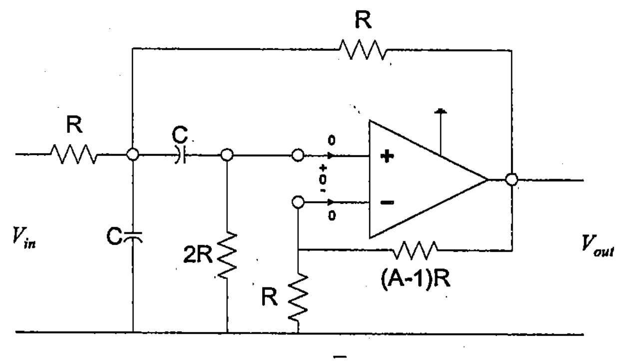
$$\omega_c = \frac{1}{RC},\quad A_v = A,\quad Q = \frac{1}{3-A}$$

S.K. High-Pass Filter



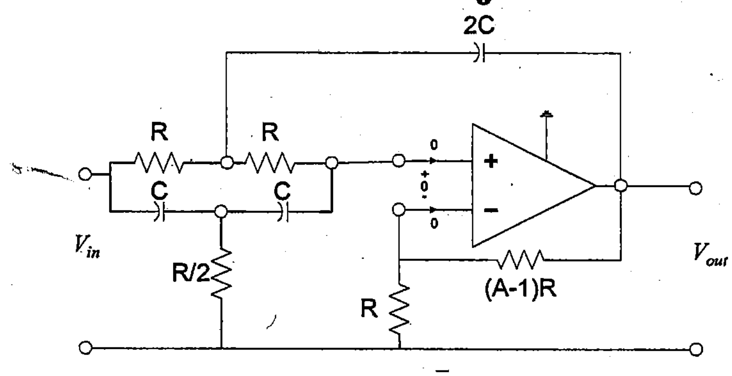
$$\omega_c = \frac{1}{RC},\quad A_v = A,\quad Q = \frac{1}{3-A}$$

S.K. Band-Pass Filter



$$\omega_o = \frac{1}{RC}\quad Q = \frac{1}{3-A}\quad A_v = AQ$$

S.K. Band-Reject Filter



$$\omega_o = \frac{1}{RC}\quad Q = \frac{1}{4-2A}\quad A_v = A$$

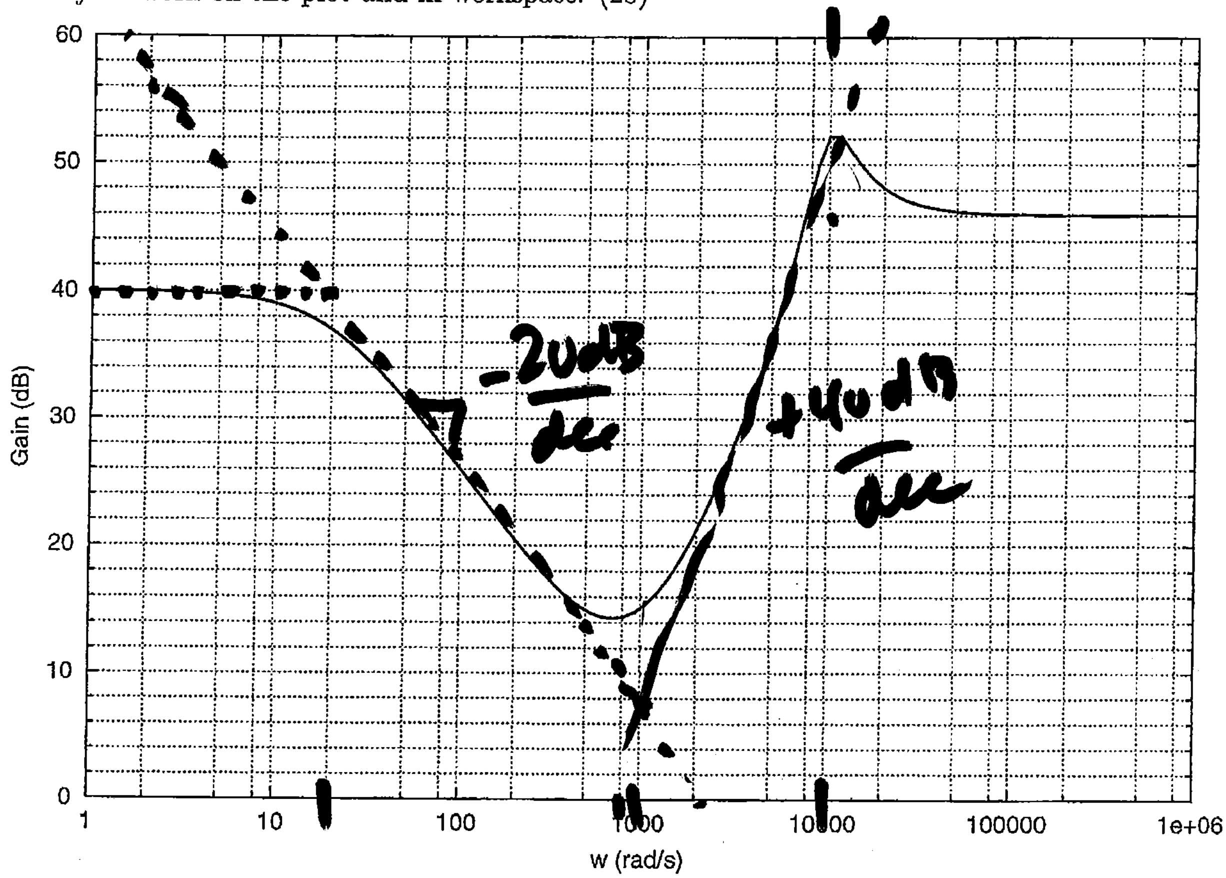
Properties of Sallen-Key Filters:

1. **Simplicity of the design**
2. **Non-Inverting Amplifier (positive Gain)**
3. **Replication of elements**

Limitations of Sallen-Key Filters:

1. **The Gain and Q are related**
2. ***Q* must be > ½, since *A* must be > 1**

5. Determine the s-domain transfer function associated with Bode magnitude plot below. Show your work on the plot and in workspace. (25)



pole at 20 rad/s

3 zeroes @ 1000 rad/s

2 poles @ 10^4 rad/s

$$H(s) = \frac{A(s+1000)\left(s^2 + \frac{1000}{Q_z}s + 10^6\right)}{(s+20)\left(s + \frac{10^4}{Q_p} + 10^8\right)}$$