Filter Design

1) Determine desired specifications (LPF, HPF, BPF etc.
2) Choose $H(s)$ to meet specs (plot)
3) Break into $1^{st}/2^{nd}$ order terms
4) Choose circuits to match each term and cascade (or sum for band reject)
5) Correct gain, scale component values (magnitude, frequency) as needed
6) Check final result

Ideal Filter

$$\cancel{H(s) = \frac{w_c}{s^n + w_c}}$$

$$\cancel{H(jw) = \frac{1}{1+\left(\frac{jw}{w_c}\right)^n}}$$

Butterworth Filter

$$|H(jw)| = \frac{1}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

- cutoff frequency $\Rightarrow \omega_c$
- $n \to \infty$ $|H(j\omega)| \Rightarrow \begin{cases} 1 & \omega < \omega_c \\ 0 & \omega > \omega_c \end{cases}$

$$|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega)$$

$$\frac{1}{1+\left[\left(\frac{\omega}{\omega_c}\right)^2\right]^n} = H(j\omega)H(-j\omega)$$

$s = j\omega \quad \omega^2 = -s^2$

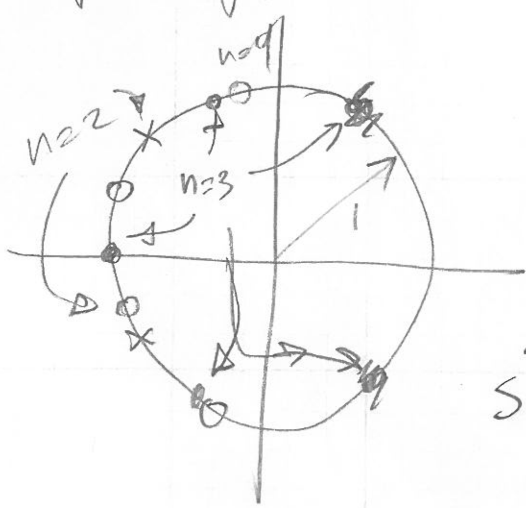
$$\frac{1}{1+\left[\left(\frac{s}{w_c}\right)^2\right]^n(-1)^n} = H(s)\,H(-s) \overset{\omega_c \to 1}{=} \frac{1}{1+(-1)^n s^{2n}}$$

$D(s)D(-s) = 1+(-1)^n s^{2n}$

$$H(s) = \frac{1}{D(s)}$$

To find $H(s)$
roots in right half plane go to $H(-s)$
left half plane go to $H(s)$

$n=1$	$D(s) = (s+1)$
$=2$	$(s^2+\sqrt{2}s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+.765s+1)(s^2+1.848s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
⋮	⋮



$$s^{2n} = (-1)^{n+1} = \begin{cases} 1\angle 180 = 1\angle 540 = \cdots & n \text{ even} \\ 1\angle 360 = 1\angle 0° & n \text{ odd} \end{cases}$$

Scaling 1Ω, $1F$, $1H$ convenient for analysis, but impractical $1F$, $1H$ too big

$\omega = \frac{1}{RC}$ or $\frac{1}{\sqrt{LC}}$ or $\frac{R}{L}$ ~ 1 rad/s

Solution: scaling

1) Magnitude scaling (k_m)   V, I divides unchanged

$R' = R * k_m$, $L' = L * k_m$, $C' = C / k_m$ $\frac{V}{I}$ $\frac{I}{V}$ $\propto$ $\frac{1}{k_m}$

$Z_{R'} = Z_R * k_m$, $Z_L * k_m$, $Z_C * k_m$

(R) (sL) $(\frac{1}{sC})$

2) Frequency scaling

$R' = R$ $L' = \frac{L}{k_f}$ $C' = \frac{C}{k_f}$

$\frac{1}{R'C'} = k_f \frac{1}{RC}$

$\frac{1}{\sqrt{L'C'}} = k_f \frac{1}{\sqrt{LC}}$

$\frac{R'}{L'} = k_f \frac{R}{L}$

General $R' = k_m R$

$C' = C / (k_m k_f)$

$L' = k_m L / k_f$

Cascading:

Using Op-amp filters, there is no loading effect — Adding R_L has no effect

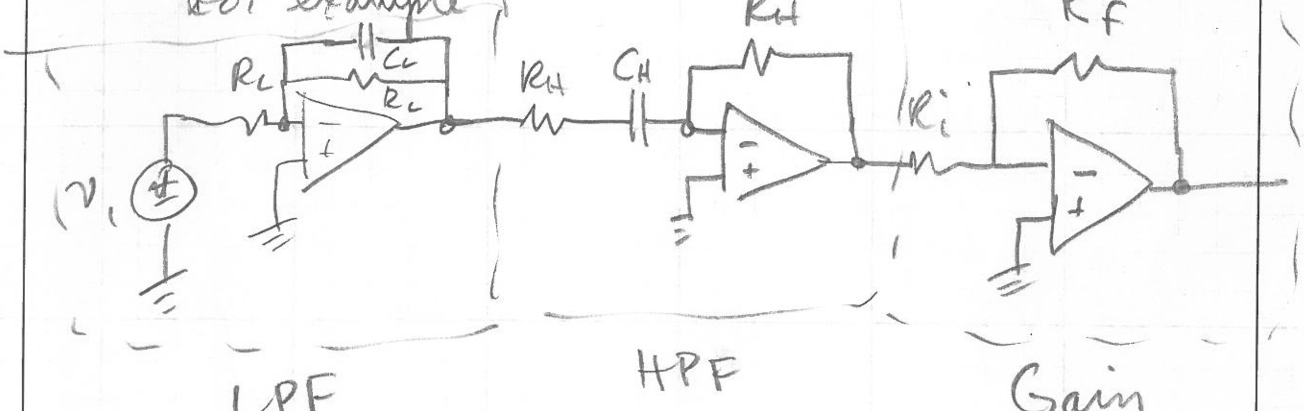
— Output is ideal voltage source (no R_s)

Thus can cascade multiple stages

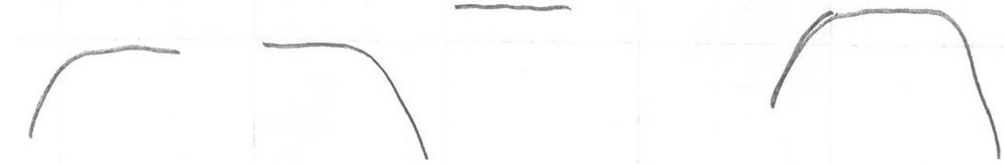
$$\to \boxed{H_1(s)} \to \boxed{H_2(s)} \to \boxed{H_3(s)} \to$$

$$H(s) = H_1(s)\,H_2(s)\,H_3(s)$$

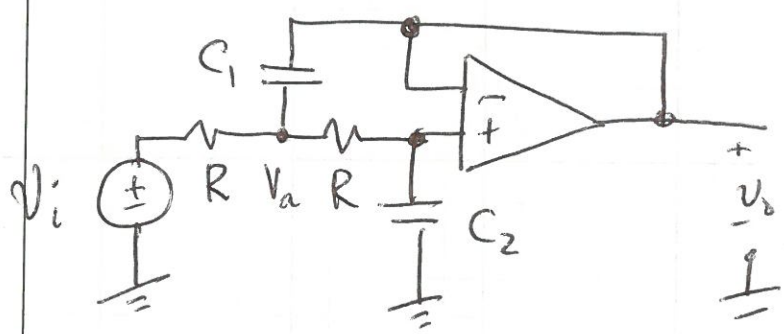
For example



$$\to \boxed{LPF} \to \boxed{HPF} \to \boxed{Gain} \to$$



2nd order Filter



$$\frac{V_i - V_a}{R} + \frac{V_o - V_a}{1/sC_1} + \frac{V_a - V_o}{R} = 0$$

$$V_o = \frac{1/sC_2}{R + 1/sC_2} V_a$$

$$\frac{V_o}{V_i} = \frac{\frac{1}{R^2 C_1 C_2}}{s^2 + \frac{2}{RC_1} s + \frac{1}{R^2 C_1 C_2}}$$

Let $R = 1\,\Omega$ (can scale later)

$$\frac{V_o}{V_i} = \frac{1/C_1 C_2}{s^2 + \frac{2}{C_1} s + \frac{1}{C_1 C_2}} = \overbrace{\frac{1}{s^2 + \frac{1}{Q} s + 1}}^{\text{Butterworth filter}}$$

$$\frac{1}{C_1 C_2} = 1 = \omega_0^2 \qquad \frac{2}{C_1} = \frac{\omega_0}{Q} = \frac{1}{Q} \qquad \omega_0 = 1$$

2nd order : $s^2 + \sqrt{2} + 1 \Rightarrow C_1 = \sqrt{2}\,F$

$C_2 = \frac{1}{\sqrt{2}} F$

But, we want cutoff frequency of

~~20,000~~ 10^5 rad/s (now 1 rad/s)

$k_f = 10^5$

$C_1 \Rightarrow C_1/10^5 = 14.14\,\mu F$

$C_2 \Rightarrow C_2/10^5 \Rightarrow 7.07\,\mu F$

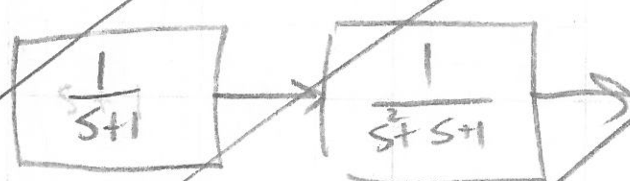
$R_g = 1\,\Omega$ still too small (currents too large

$k_m = 10^4 \Rightarrow R \Rightarrow k_m 1\,\Omega = 10k\Omega$

$C_1 \Rightarrow C_1/k_m = 1.41\,nF$

$C_2 \Rightarrow C_2/k_m = 0.71\,nF$

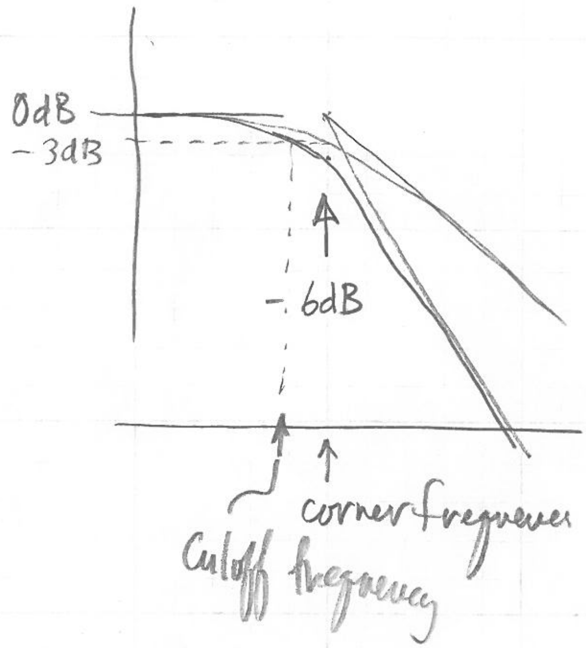
Cascade to get higher order filter



3rd order Butterworth LPF

Higher order filters (e.g. 2nd order LPF)

$H_1(s) = \frac{-\omega_n}{s+\omega_n}$

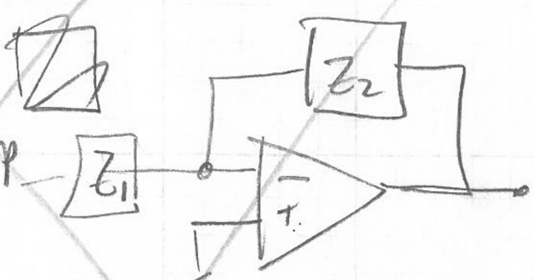


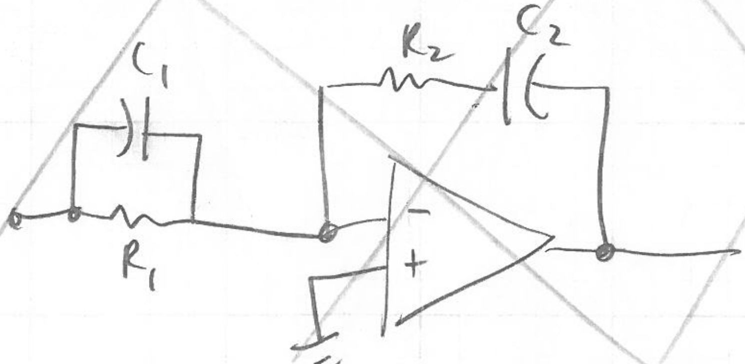
$$H(s) = \frac{(-1)^n \omega_n^n}{(s+\omega_n)^n}$$

$$H(j\omega) = \frac{(-1)^n}{\left(1+j\frac{\omega}{\omega_n}\right)^n}$$

$$|H(j\omega)| = \frac{1}{\left[\sqrt{1+(\omega/\omega_n)^2}\right]^n}$$

~~Wilaba~~ Low Q (high Q w/L) 2nd order or higher circuits can use simple inverting or non-inverting op-amp structures





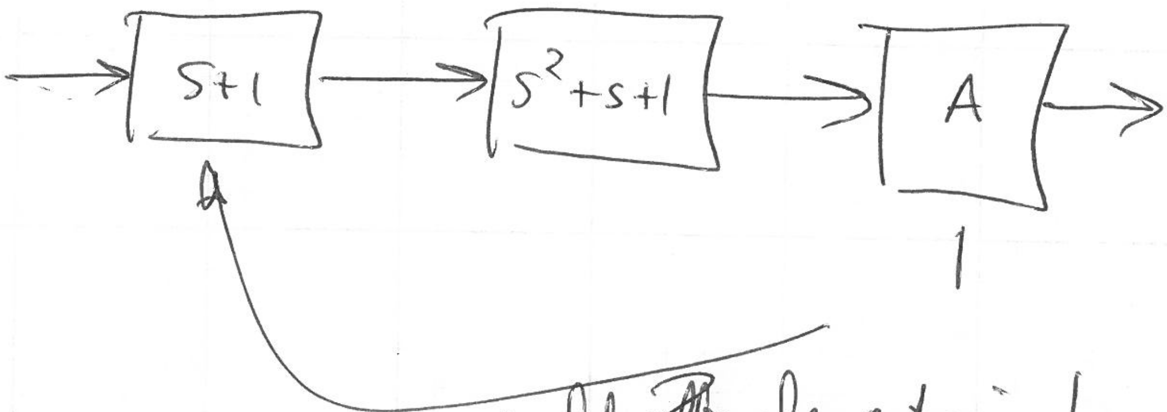
$$\frac{V_o}{V_i} = \frac{R_2 + 1/sC_2}{1/(1/R_1 + sC_1)}$$

$$= (R_2 + 1/sC_2)(1/R_1 + sC_1)$$

$$= \frac{1}{s^2 C_1 C_2}(1+sR_1C_1)(1+sR_2C_2)$$

Higher order filters

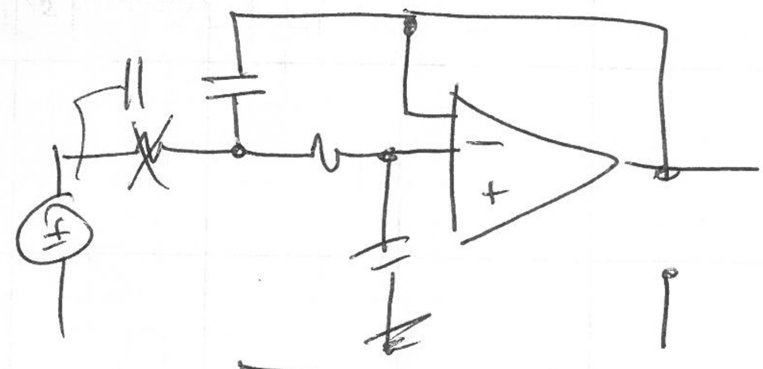
Gain (if desired)



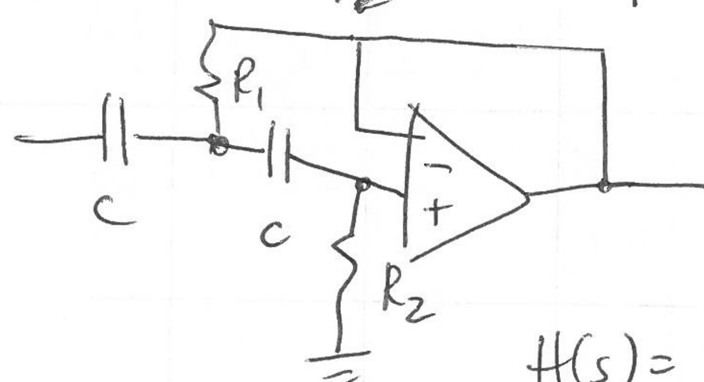
could ~~als~~ also put gain here

HPF : reverse R & C

LPF



HPF

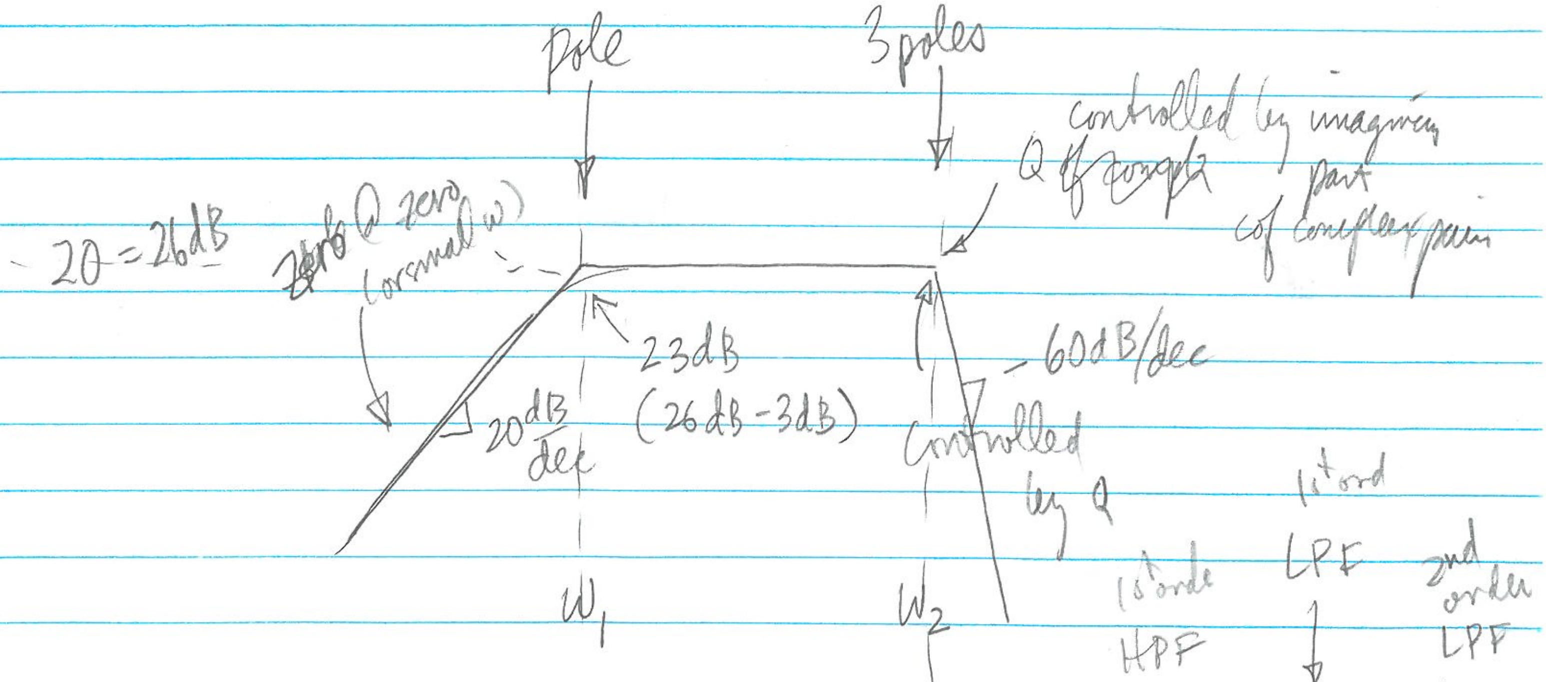


$$H(s) = \frac{s^2}{s^2 + \frac{2}{R_2 C}s + \frac{1}{R_1 R_2 C^2}} = \frac{s^2}{s^2 + \frac{s}{Q} + 1}$$

Let $C = 1F$

$\frac{1}{R_1 R_2} = 1$, $b_1 = 2/R_2$

Example: BPF



1st order HPF

1st ord LPF

2nd order LPF

$$H(s) = \frac{As}{(s+\omega_1)(s+\omega_2)(s^2+\frac{\omega_2}{Q}+\omega_2^2)} = 20\left(\frac{s}{s+\omega_1}\right)\left(\frac{\omega_2}{s+\omega_2}\right)\left(\frac{\omega_2^2}{s^2+\frac{\omega_2}{Q}+\omega_2^2}\right)$$

$\omega_1 \ll \omega \ll \omega_2$ $\quad H(s) \cong \dfrac{\cancel{s}}{\cancel{(s)}(\omega_2)(\omega_2^2)} = \dfrac{1}{\omega_2^3}$

$A = 20\,\omega_2^3$

$Q =$

Required filter order

Butterworth Filter

~~A(w) = 20~~ $|H(j\omega)| = \dfrac{1}{\sqrt{1+(\omega/\omega_c)^{2n}}}$

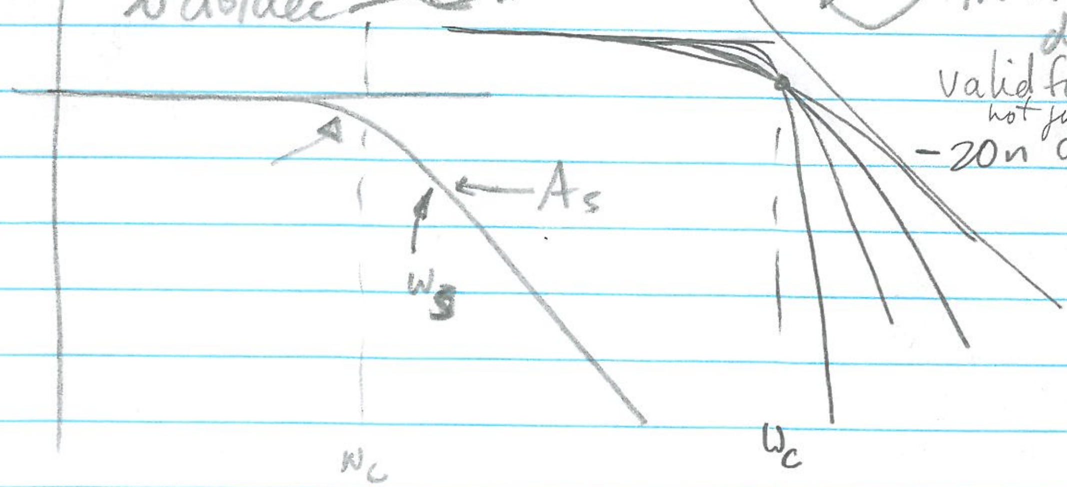
$A_{dB}(\omega) = 20\log_{10}\dfrac{1}{\sqrt{1+(\omega/\omega_c)^{2n}}}$

$= -10\log_{10}\left(1+\left(\omega/\omega_c\right)^{2n}\right) \Longrightarrow$ specify greater than or less than design value

For $\left(\dfrac{\omega}{\omega_c}\right)^{2n} >> 1$ $\quad A_{dB} \cong -20n\log_{10}\left(\dfrac{\omega}{\omega_c}\right)$

for $A < A_s$ $\quad n > \dfrac{-A_s}{20\log_{10}(\omega/\omega_c)}$ fraction of a decade

20 dB/dec



Otherwise $\quad A = -10\log_{10}\left[1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}\right]$

$10^{-\frac{A}{10}} = 1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}$

$\left(\dfrac{\omega}{\omega_c}\right)^{2n} = \left(10^{-A/10}-1\right)$

$2n\log_{10}\left(\dfrac{\omega}{\omega_c}\right) = \log_{10}\left(10^{-A/10}-1\right)$

$n = \dfrac{\log_{10}\left(10^{-A/10}-1\right)}{2\log_{10}(\omega/\omega_c)}$ use next higher integer for n

Butterworth
For HPF

$$|H(j\omega)| = \frac{(\omega/\omega_c)^n}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

$$= \frac{1}{\sqrt{1+\left(\frac{\omega_c}{\omega}\right)^{2n}}}$$

$$\frac{\omega}{\omega_c} \rightarrow \frac{\omega_c}{\omega}$$

$$n = \frac{\log_{10}(10^{-A/10}-1)}{2\log_{10}(\omega_c/\omega)}$$

Example HPF w/ $\omega_c = 10^3$ rad/s, $A < -30\text{dB}$ @ $200 \frac{\text{rad}}{\text{s}}$, $A > -1\text{dB}$ @ $2000 \frac{\text{rad}}{\text{s}}$

$$n = \frac{\log_{10}(10^3-1)}{2\log_{10}(5)} = \frac{3.00}{2(0.7)} = \frac{3}{1.4} = 2.14$$

$$\rightarrow n \geq 2.14 \quad n=3$$

$$n = \frac{\log_{10}(10^{0.1}-1)}{2\log_{10}(1/2)} = \frac{\log_{10}(0.259)}{2\log(0.5}$$

$$= \frac{-0.587}{2(-0.3)} = 0.98 \quad n=1$$

Quick test $n \geq \frac{-(-30)}{20\log_{10}(5)} = 2.14$