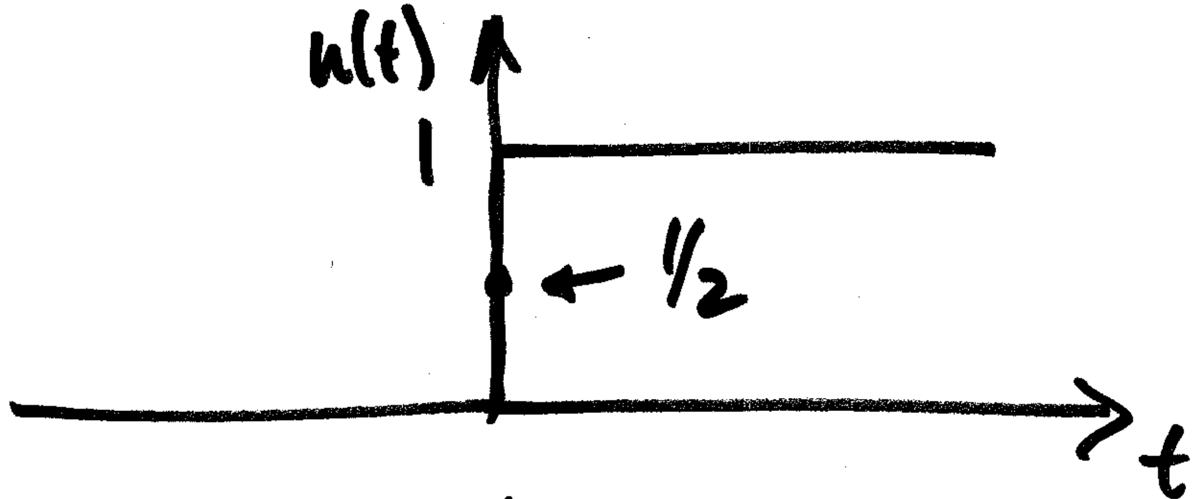
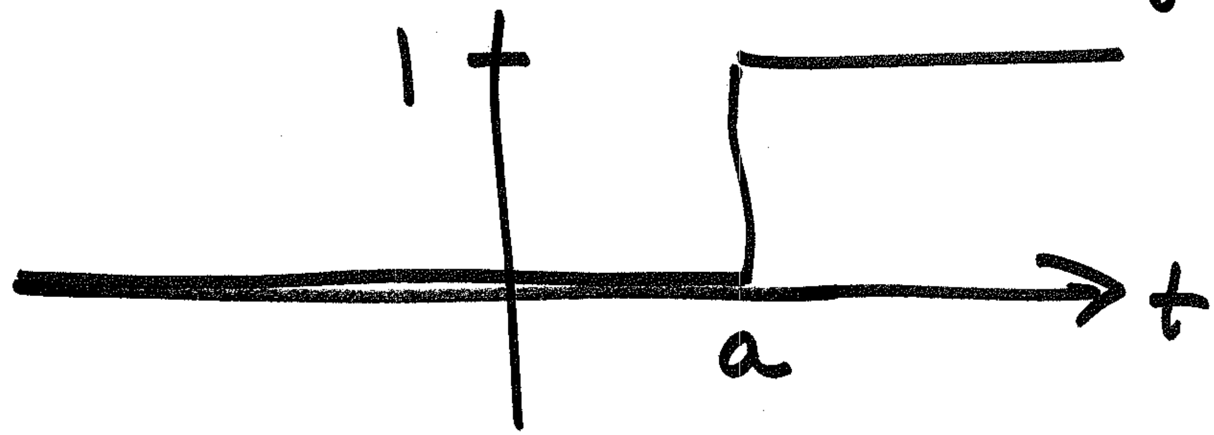
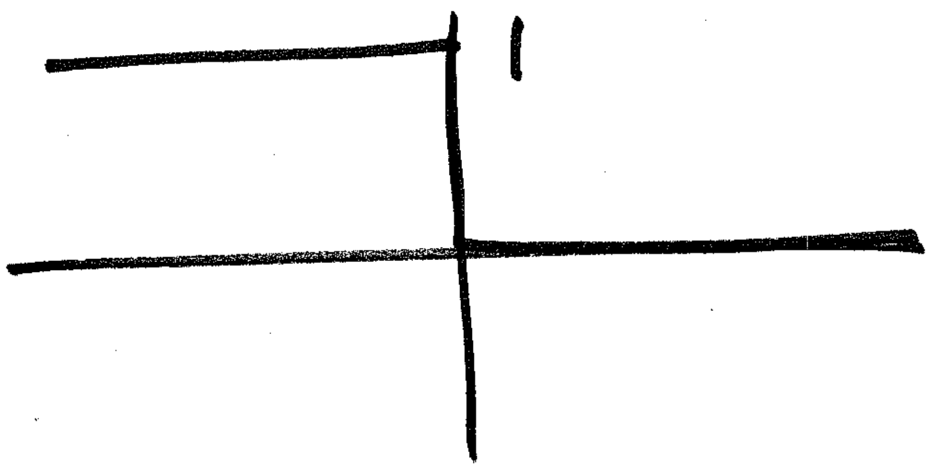
$u(t)$



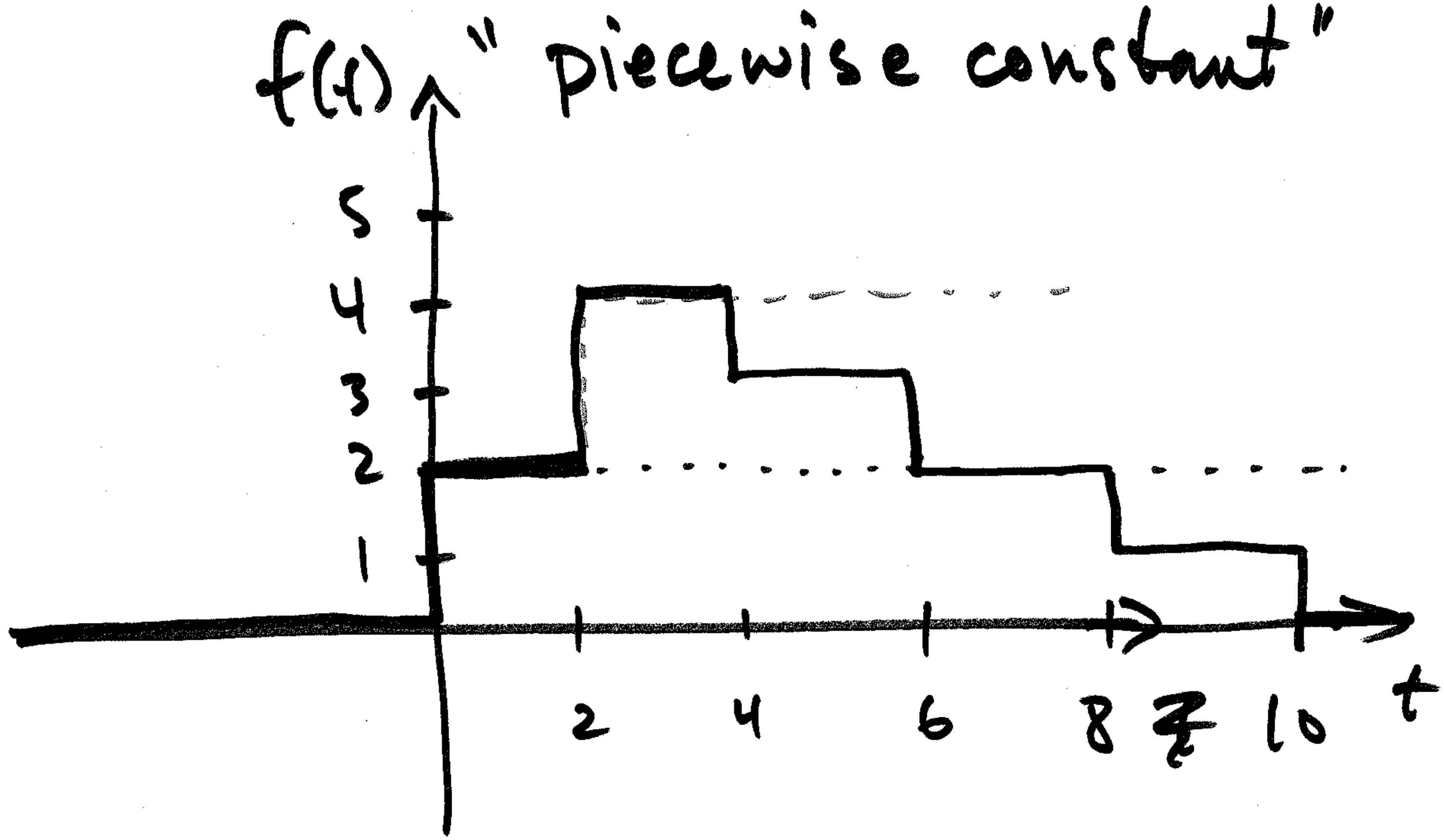
$u(t-a)$
"shifted"



$u(-t)$



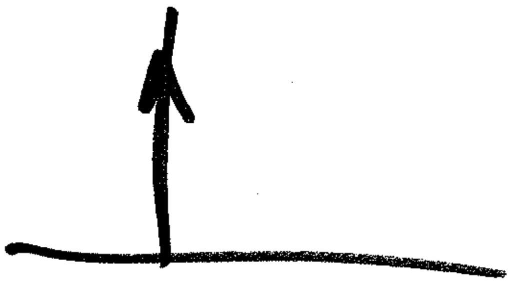
"piecewise constant"

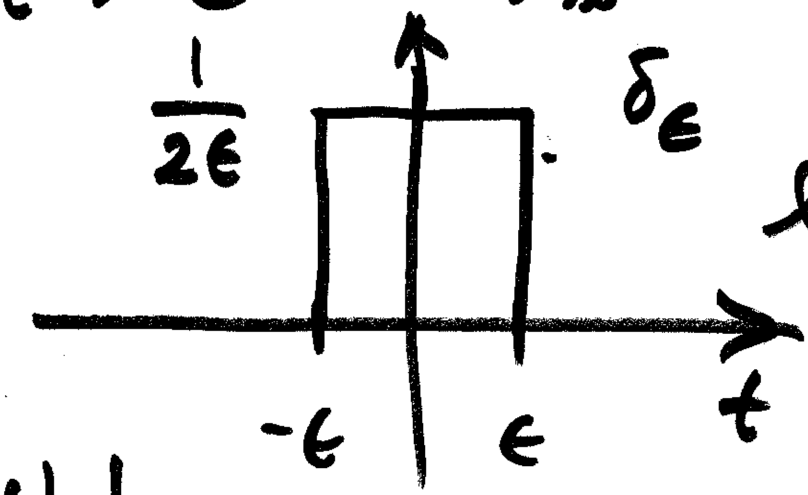


$$f(t) = 2u(t) + 2u(t-2) - u(t-4) - u(t-6) - u(t-8) - u(t-10)$$

Impulse $\delta(t)$ Dirac delta function

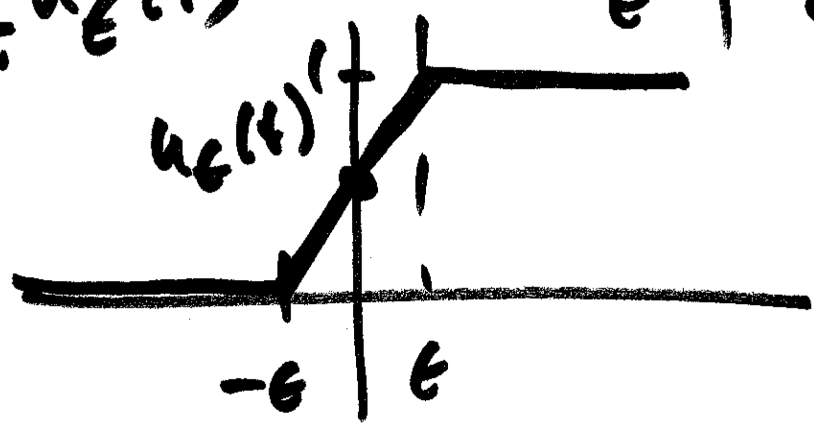
$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



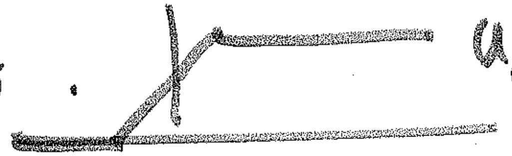


$$\lim_{\epsilon \to 0} \delta_\epsilon = \delta$$

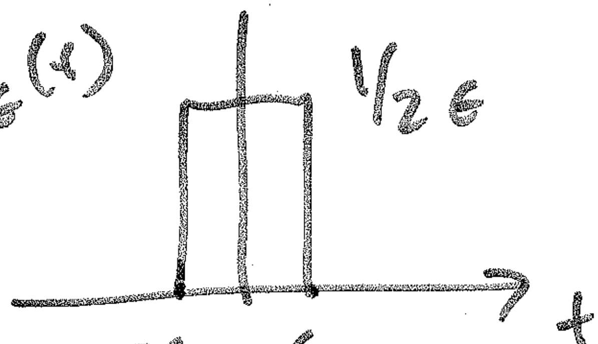
$$\delta_\epsilon(t) = \frac{d}{dt} u_\epsilon(t)$$



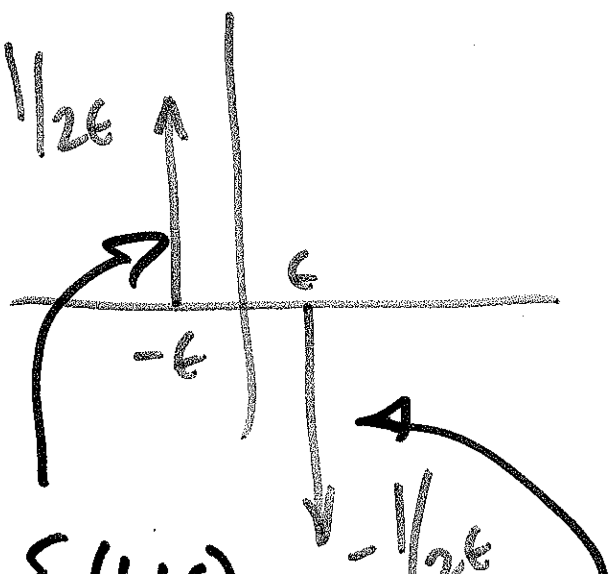
$\int_0^t \delta_\epsilon(t)\,dt$



$\delta_\epsilon(t)$

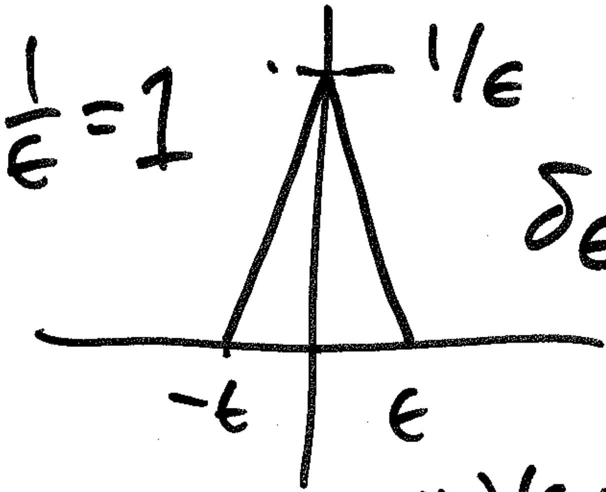


$\delta'_\epsilon(t)$



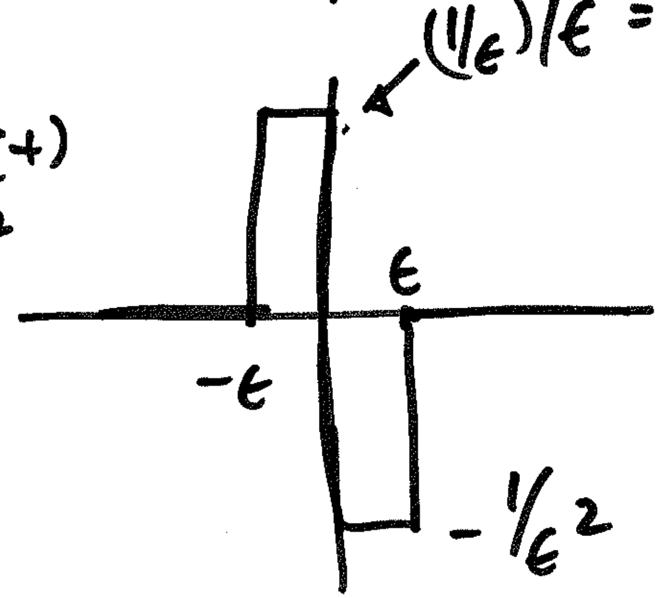
$$\frac{1}{2\epsilon}\delta_\epsilon(t+\epsilon) + \frac{1}{2\epsilon}\delta_\epsilon(t-\epsilon)$$

$$\frac{1}{2}2\epsilon\frac{1}{\epsilon}=1$$

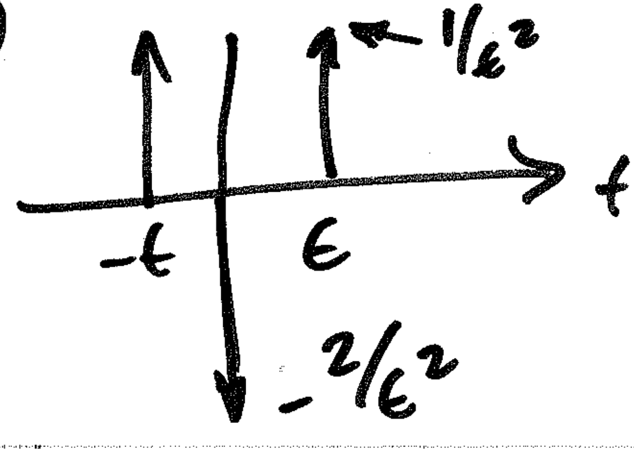


$\delta_{\epsilon_2}(t)$

$\delta'_{\epsilon_2}(t)$



$\delta''_{\epsilon_2}(t)$



$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = \int_{0^-}^{0^+} \delta(t) e^{-st}\, dt$$

$$= \int_{0^-}^{0^+} \delta(t)\, \overset{1}{\cancel{e^{0}}}\, dt = 1$$

Sifting property of δ

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a)$$

$$= \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} \delta(t-a) f(a)\, dt = f(a)$$

$$\mathcal{L}\{\delta'(t)\} = \int_{0^-}^{\infty} \delta'(t) e^{-st}\,dt = \lim_{\epsilon\to 0} \int_{-\epsilon}^{\infty} \delta_\epsilon'(t) e^{-st}\,dt$$

$$= \lim_{\epsilon\to 0} \left[\int_{-\epsilon}^{\infty} \left\{ \frac{\delta(t+\epsilon)}{2\epsilon} - \frac{\delta(t-\epsilon)}{2\epsilon} \right\} e^{-st}\,dt \right]$$

sifting

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \left[e^{-s(-\epsilon)} - e^{-s(\epsilon)} \right]$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \left[(\cancel{1} + s\epsilon + \cancel{O(\epsilon^2)}) - (\cancel{1} - s\epsilon + \cancel{O(\epsilon^2)}) \right]$$

$$= \lim_{\epsilon\to 0} \frac{2s\epsilon}{2\epsilon} = \underline{\underline{s}}$$

$$-\left.\frac{d(e^{-st})}{dt}\right|_{t=0}$$

Functional Transforms

(a) $\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt = \frac{1}{s}$

(b) $\mathcal{L}\{e^{-at}\} = \int_{0^-}^{\infty} e^{-at} e^{-st} dt$

$$= \int_{0^-}^{\infty} e^{-(s+a)t} dt$$

$$= \frac{-1}{s+a} e^{-(s+a)t} \Big|_{0^-}^{\infty} = \frac{-1}{s+a}(0-1)$$

$$= \frac{1}{s+a}$$

$$\mathcal{L}\{\cos \omega t\} = \int_{0^-}^{\infty} \cos(\omega t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] e^{-st}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{s+j\omega} + \frac{1}{s-j\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s-\cancel{j\omega}}{s^2+\omega^2} + \frac{s+\cancel{j\omega}}{s^2+\omega^2}\right]$$

$$= \frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right) = \frac{s}{s^2+\omega^2}$$

$\delta'(t)$ s

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \underbrace{\frac{df}{dt}\,dt}_{dv=df}\;\overset{u}{e^{-st}} = \int_{t=0^-}^{t=\infty} u\,dv$$

$$= uv - \int v\,du$$

$$= e^{-st}f(t)\Big|_{0^-}^{t=\infty} - \int_{t=0^-}^{t=\infty} \underbrace{f(t)}_{v}\overbrace{(-s)e^{-st}\,dt}^{du}$$

$$= \left[0 - f(0^-)\right] - (-s)\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$= s\,F(s) - f(0^-)$$

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dfrac{df(0^-)}{dt}$
nth derivative (time)	$\dfrac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dfrac{df(0^-)}{dt} - s^{n-3}\dfrac{df^2(0^-)}{dt^2} - \cdots - \dfrac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\displaystyle\int_0^t f(x)\,dx$	$\dfrac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\dfrac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
s integral	$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(u)\,du$

Example $\mathcal{L}\left\{t^2 e^{-at}\right\}$

$$f(t) = e^{-at} \iff F(s) = \frac{1}{s+a} \quad (FT)$$

$$g(t) = t^2 e^{-at}$$

$$= t^2 f(t) \iff (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+a}\right)$$

$$= \frac{d}{ds}\left(-\frac{1}{(s+a)^2}\right) \quad (OT)$$

$$t^2 e^{-at} \iff \frac{2}{(s+a)^3}$$

$$f(t) \iff F(s)$$