Homework #3 Solutions

1 (a) $N_d = n = \int_{-\infty}^{\infty} f_{FD}(E)\left[N_c(E) + N_{ib}(E)\right] dE$

at 0K $f_{FD}(E) = \begin{cases} 1 & E < E_f \\ 0 & E > E_f \end{cases}$

so $N_d = 4\times10^{19} = \int_{-\infty}^{E_f} \left(N_c(E) + N_{ib}(E)\right) dE$

~~Try assuming $E_f < E_c$~~ ✓

$n = \int_{E_c - 0.09eV}^{E_f} N_{ib}(E)\, dE$

$4\times10^{19}\,\text{cm}^{-3} = \frac{\cancel{4\pi}\times10^{19}\,\text{cm}^{-3}}{\cancel{(0.12eV)}}\left(\frac{\cancel{0.12eV}}{\cancel{\pi}}\right) \sin\left[\frac{\pi(E - E_c + 0.03eV)}{0.12eV}\right]_{E_c - 0.09}^{E_f}$

$4\times10^{19} = 4\times10^{19}\,\text{cm}^{-3}\left[\sin\left(\frac{\pi\,(E_f - E_c + 0.03eV)}{0.12eV}\right) + 1\right]$

$\Rightarrow E_f = E_c - 0.03eV$ (Assumption checks)

(b) $E_f = E_c - 0.03eV$, $E_f >> E_v$, so MB approx good in V-band.

$p = N_v \exp\left(-\frac{E_f - E_v}{kT}\right) = 2.5\times10^{19}\,\text{cm}^{-3} \exp\left(-\frac{1.12eV - 0.03eV}{kT}\right) = 13\ \text{cm}^{-3}$

$np = (4\times10^{19}\,\text{cm}^{-3})(13\,\text{cm}^{-3}) = 5.2\times10^{20}\,\text{cm}^{-6} = 5.2\times n_i^2$, $\Delta E \sim 0.04\,eV$

Other effects (rigid band gap shift, band tailing) further reduce E_g^{eff} increasing ΔE, n_{ie}^2

$$C_{ox}' = \frac{K_{ox}\epsilon_o}{x_{ox}} = \frac{(3.9)(8.854\times10^{-14}\,F/cm)}{60\times10^{-8}\,cm} = 5.76\times10^{-7}\,F/cm^2$$

2. $\phi_{ms} = \frac{1}{q}(E_f - E_c)_s - \frac{1}{q}(E_f - E_c)_m$

$$= -\frac{E_g}{q} + \frac{kT}{q}\ln\left(\frac{N_v}{N_a}\right) - 0.05\ V = -1.12eV + kT\ln\left(\frac{1.8\times10^{19}}{5\times10^{17}}\right) - 0.05$$

$$= -1.08\,V$$

$$V_{FB} = \phi_{ms} - \frac{Q_{ss}'}{C_{ox}'} = -1.08\,V - \frac{(1.6\times10^{-19}c)(5\times10^{10}\,cm^{-2})}{5.76\times10^{-7}\,F/cm^2} = -1.09\,V$$

(a) $V_{gb} = -1V > V_{FB} = -1.09V \Rightarrow$ depletion

$$V_{gb} = \phi_{ms} + \psi_{ox} + \psi_s = V_{FB} + \Delta\psi_{ox} + \psi_s$$

$$= -1.09V - \frac{Q_s'}{C_{ox}'} + \psi_s$$

$$Q_s' = -\sqrt{2qN_aK_s\epsilon_o\psi_s} = -4.07\times10^{-7}\frac{C}{V^{1/2}cm^2}\sqrt{\psi_s} = -4.5\times10^{-8}\,C/cm^2$$

$$V_{gb} = -1.09V + 0.71\sqrt{\psi_s} + \psi_s = -1V \Rightarrow \underline{\underline{\psi_s = 0.012V}}$$

$$Q_g = -(Q_s' + Q_{ss}') = -\left(-4.5\times10^{-8}\,C/cm^2 + 8\times10^{-9}\,C/cm^2\right)$$

$$= \underline{\underline{3.7\times10^{-8}\,C/cm^2}}$$

$$\Delta\psi_{ox} = 0.71\sqrt{\psi_s} = 0.08\,V \qquad \psi_{ox} = -\frac{Q_{ss}'}{C_{ox}'} + \Delta\psi_{ox} = \underline{0.07\,V}$$

$$\psi_B = \frac{kT}{q}\ln\left(\frac{N_a}{n_i}\right) = 0.45\,V$$

(b) $V_T = V_{FB} - \frac{Q_d'^{max}}{C_{ox}'} + 2\psi_B$

$$Q_d'^{max} = -4.07\times10^{-7}\frac{C}{V^{1/2}cm^2}\sqrt{\psi_s} = -C_{ox}'\,0.71\sqrt{0.9V}\,V^{1/2}$$

$$V_T = -1.09V + 0.71\sqrt{0.9} + 0.9 = 0.48V$$

$V_{gb} = 0V < V_T \Rightarrow$ still depletion

$V_{gb} = -1.09 + 0.71\sqrt{\psi_s} + \psi_s = 0 \qquad \underline{\underline{\psi_s = 0.56V}}$

$Q_g' = -(Q_s' + Q_{ss}') = \underline{\underline{3.0 \times 10^{-7} \, C/cm^2}}$

$\psi_{ox} = V_{gb} - \phi_{ms} - \psi_s = 0 + 1.08 - 0.56 = \underline{\underline{0.52V}}$

(c) $V_{gb} = 2V > V_T \Rightarrow$ inversion

Neglecting any voltage required to form inversion layer once $n_s = p_{bulk}$:

$$\psi_s = 2\psi_B = 0.9V$$

$\psi_{ox} = V_{gb} - \phi_{ms} - \psi_s = 2.0 + 1.08 - 0.9 = 2.18V$

$Q_g' = C_{ox}' \psi_{ox} = 1.26 \times 10^{-6} \, C/cm^2$

More accurately, in inversion $\psi_s \cong 2\psi_B + 6kT/q = 1.06V$

Then $\psi_{ox} = 2.02V$ and $Q_g' = 1.16 \times 10^{-6} \, C/cm^2$

It is also possible to use result from text that

$$Q_s = -\sqrt{2qK_s\epsilon_o N_a}\left(\psi_s + \frac{kT}{q}\frac{n_i^2}{N_a^2}e^{q\psi_s/kT}\right)^{1/2}$$

$$= -4.07 \times 10^{-7} \frac{C}{cm^2 V^{-1/2}}\left[\psi_s + 2 \times 10^{-17} V e^{q\psi_s/kT}\right]^{1/2}$$

$V_{gb} = V_{FB} + 0.71\left[\psi_s + 2 \times 10^{-17} e^{q\psi_s/kT}\right]^{1/2} + \psi_s = 2V \Rightarrow \psi_s = 1.05V$

Very close to 2nd estimate

4 (a) In VERY strong inversion, the $e^{q\psi_s/kT}$ term dominates in Equation 2.154 (4th term)

$$Q_I' \cong Q_s' \cong \sqrt{2K_s\epsilon_0 kT\,(n_i^2/N_d)}\,\exp\left(-q\psi_s/2kT\right)$$

With poly depletion, $\psi_{poly} = -\dfrac{Q_{poly}^2}{2K_s\epsilon_0 q N_{poly}} = -\dfrac{Q_s^2}{2K_s\epsilon_0 q N_{poly}}$

$$V_{gb} = \phi_{ms} + \psi_{poly} + \psi_{ox} + \psi_s \qquad \psi_{ox} = -\frac{Q_s'}{C_{ox}'}$$

$$\frac{dQ_s'}{dV_{gb}} = \frac{dQ_s'}{d\psi_s} \Big/ \frac{dV_{gb}}{d\psi_s}$$

$$\frac{dQ_s'}{d\psi_s} = -\frac{q}{2kT}Q_s' \qquad \frac{dV_{gb}}{d\psi_s} = 1 - \left[\frac{2Q_s}{2K_s\epsilon_0 q N_{poly}} + \frac{1}{C_{ox}'}\right]\frac{dQ_s}{d\psi_s}$$

$$\frac{dQ_s'}{dV_{gb}} = \frac{-\frac{q}{2kT}Q_s'}{1 + \frac{q}{2kT}Q_s'\left[\frac{1}{C_{ox}'} + \frac{Q_s'}{K_s\epsilon_0 q N_{poly}}\right]}$$

$$= -\frac{1}{\frac{2kT/q}{Q_s'} + \frac{1}{C_{ox}'} + \frac{Q_s'}{K_s\epsilon_0 q N_{poly}}}$$ see eq. 2.185 in text

$$C_{ox}' = 1.15\times10^{-6}\ \mathrm{F/cm^2}$$

(b) In inversion, $Q'_{I0} \cong -C'_{ox}(V_{GB}-V_T)$ without polydepletion or x_I

$$Q'_{I0} \cong \begin{cases} 3.45\times10^{-7}\ C/cm^2 & V_{gb}=V_T-0.3V \\ 1.15\times10^{-6}\ C/cm^2 & =V_T-1V \end{cases}$$

$$\frac{2kT/q}{Q'_S} \cong \begin{cases} 1.5\times10^{5}\ cm^2/F & V_T-0.3V \\ 4.5\times10^{4}\ cm^2/F & V_T-1V \end{cases}$$

$$\frac{Q'_S}{K_s\epsilon_0 q N_{poly}} = \begin{cases} 5.2\times10^{4}\ cm^2/F & V_T-0.3V \\ 1.7\times10^{5}\ cm^2/F & V_T-1V \end{cases}$$

$$1/C'_{ox} = 8.7\times10^{5}\ cm^2/F$$

For $V_{gb}-V_T=-1V$, poly depletion is most important. Inversion layer thickness is significant for lower overdrive (smaller Q'_S).

Due to X_{inv} $\frac{dQ'_S}{dV_{GB}}$ reduced by $\begin{cases} 0.85 = \frac{8.7\times10^5}{1.5\times10^5+8.7\times10^5} & V_T-0.3 \\ 0.95 & V_T-1V \end{cases}$

Due to X_{poly} $\frac{dQ'_S}{V_{GB}}$ reduced by $\begin{cases} 0.94 & V_T-0.3V \\ 0.84 & V_T-1V \end{cases}$

For both, $\frac{dQ_s}{V_{GB}}$ reduced by $\begin{cases} 0.81 & V_T-0.3V \\ 0.80 & V_T-1V \end{cases}$

For the actual drop in total Q_s due to these effects, we need to solve

$$\psi_B = -\frac{kT}{q}\ln\left(\frac{N_d}{n_i}\right)$$

$$V_{gb} = \Phi_{ms} + \psi_{poly} + \psi_{ox} + \psi_s$$

$$= \Phi_{ms} + \frac{-Q_s'^2}{2K_s\epsilon_0 q N_{poly}} - \frac{Q_s'}{C_{ox}'} - \frac{2kT}{q}\ln\left[\frac{Q_s'}{\sqrt{2K_s\epsilon_0 kT(n_i^2/N_d)}}\right]$$

$$\cong V_T - 2\psi_B - \frac{Q_I'^2}{2K_s\epsilon_0 q N_{poly}} - \frac{Q_I'}{C_{ox}'} - \frac{2kT}{q}\ln\left[\quad\right]$$

$$V_{gb} - V_T = \frac{-Q_I'^2}{1.34\times10^{-11}\ \frac{CF}{cm^4}} - \frac{Q_I'}{1.15\times10^{-6}\frac{F}{cm^2}} - 0.052V\ln\left[\frac{Q_I'}{9.3\times10^{-8}\frac{C}{cm^2}}\right]$$

Including just poly depletion (first two terms), can solve quadratic

$$Q_I = \begin{cases} 3.35\times10^{-7} & 97\% & 0.3V \\ 1.06\times10^{-6} & 92\% & 1.0V \end{cases}$$

These are higher than change in $\frac{dQ}{dV}$ since average depth into poly is $x_d/2$

Including just x_I

$$Q_I = \begin{cases} 2.76\times10^{-7} & 80\% & 0.3 \\ 1.02\times10^{-6} & 89\% & 1.0V \end{cases}$$

These are lower as initially charge is added deeper into silicon

Now x_I dominates in both cases.

3 $\displaystyle\int_0^\infty \psi^* \psi \, dx = 1 = \int_0^\infty C^2 x^2 e^{-\gamma x} dx$ w/ $\psi = C x e^{-\frac{\gamma x}{2}}$

By parts $u = C^2 x^2$ $dv = e^{-\gamma x} dx$

$du = 2C^2 x \, dx$ $v = -\frac{1}{\gamma} e^{-\gamma x}$

$$1 = uv\Big|_0^\infty - \int_0^\infty v \, du = \cancel{\frac{-C^2 x^2}{\gamma} e^{-\gamma x}}\Big|_0^\infty \xrightarrow{} 0 - \int_0^\infty -\frac{e^{-\gamma x}}{\gamma} 2C^2 x \, dx$$

$$= \int_0^\infty \frac{2C^2 x}{\gamma} e^{-\gamma x} dx \quad \text{again by parts}$$

$$= \cancel{\left[\left(\frac{2C^2 x}{\gamma}\right)\left(-\frac{1}{\gamma} e^{-\gamma x}\right)\right]_0^\infty} \xrightarrow{} 0 - \int_0^\infty \left(-\frac{1}{\gamma} e^{-\gamma x}\right)\left(\frac{2C^2}{\gamma} dx\right)$$

$$= \frac{2C^2}{\gamma^2} \int_0^\infty e^{-\gamma x} dx = \frac{2C^2}{\gamma^3} = 1 \qquad C = \sqrt{\frac{\gamma^3}{2}}$$

(a) $$\langle E \rangle = \int_0^\infty \psi^* \left[-\frac{\hbar^2}{2m_x} \frac{\partial^2}{\partial x^2} + q\mathcal{E}x \right] \psi \, dx$$

$$= -\frac{\hbar^2}{2m_x} \int_0^\infty \psi^* \frac{\partial^2 \psi}{\partial x^2} dx + q\mathcal{E} \int_0^\infty \psi^* x \psi \, dx$$

$$= \frac{\hbar^2 \gamma^2}{8m_x} + \frac{3q\mathcal{E}}{\gamma} = \text{Kinetic energy} + \text{Potential Energy}$$

To minimize, $\frac{d\langle E \rangle}{d\gamma} = 0 \Rightarrow \gamma = \left(\frac{12 q\mathcal{E} m_x}{\hbar^2} \right)^{1/3}$

$\displaystyle\frac{2 \times \cancel{6}^{\,3} \times \cancel{4} \times 3}{\cancel{8} \times 2 \times 4 \times 8}$

$$\langle E \rangle = \left[\frac{\hbar^{\cancel{6}\,2} (12)^{\cancel{3}\,2} (q\mathcal{E})^2 \cancel{m_x^2}}{8^3 m_x^{\cancel{3}} \cancel{\hbar^4}} \right]^{1/3} + \left[\frac{\hbar^2 3^3 (q\mathcal{E})^{\cancel{3}\,2}}{12 \cancel{(q\mathcal{E})} m_x} \right]^{1/3} = A \left[\frac{\hbar^2 (q\mathcal{E})^2}{m_x} \right]^{1/3}$$

where $A = \left[\frac{9}{32}\right]^{1/3} + \left[\frac{9}{4}\right]^{1/3} = \left[\frac{9}{4}\right]^{1/3} \left(\frac{1}{2} + 1\right) = \left[\frac{3}{2}\right]^{5/3}$

(b) For Si, $m_x = m_t$ for 4 minima & $\quad m_t = 0.19 m_0$
m_l for 2 minima $\quad m_l = 0.98 m_0$

$$\langle x \rangle = \int_0^\infty \psi^* x \psi \, dx$$

$$= \frac{3}{\gamma} \quad \text{using result from (a) for potential energy}$$

For $m_x = 0.19 m_0$, $\gamma = \left(\frac{12 q \mathcal{E} m_x}{\hbar^2} \right)^{1/3}$

Let $\mathcal{E} = \alpha \frac{10^6 V}{cm}$

$1 C \cdot V = 1 J$

$1 J = \frac{kg\, m^2}{s^2}$

$$= \left[\frac{12 (1.6 \times 10^{-19} C)(\alpha 10^6 \frac{V}{cm})(0.19)(9.11 \times 10^{-31} kg)}{(1.055 \times 10^{-34} J \cdot s)^2} \right]^{1/3}$$

$$= \left[\frac{12 (1.6 \times 10^{-19})(\alpha 10^8 m^{-1})(0.19)(9.11 \times 10^{-31} \cancel{kg})}{(1.055 \times 10^{-34})^2 \cancel{kg} m^2} \right]^{1/3}$$

$$= \alpha^{1/3} (1.44 \times 10^9 m^{-1}) = \alpha^{1/3} 1.44 \times 10^7 cm^{-1}$$

$\langle x \rangle = \alpha^{-1/3} \, 2.08 \, nm$

For $m_x = 0.98 m_0$, $\langle x \rangle = \alpha^{-1/3} \, 2.08 \left(\frac{0.19}{0.98} \right)^{1/3} = \alpha^{-1/3} \, 1.20 \, nm$

Classically for constant field $n = n_s e^{-q\mathcal{E}x/kT}$,

so $\langle x \rangle = \frac{kT}{q\mathcal{E}} = \frac{0.026 V}{\mathcal{E}} = \frac{0.26}{\alpha} nm$

$$\langle E \rangle = \left[\frac{3}{2} \right]^{5/3} \left[\frac{(6.6 \times 10^{-16} eV \cdot s)^2 \frac{10^{16}}{\cancel{m^2}} \alpha^2 (eV)(1.6 \times 10^{-19} \cancel{kg\, m^2 / s^2})}{(m_x/m_0)\, 9.11 \times 10^{-31} \cancel{kg}} \right]^{1/3} = \begin{cases} 0.313 \, \alpha^{2/3} & m_x = m_t \\ 0.181 \, \alpha^{2/3} & m_x = m_l \end{cases}$$

For $\alpha \sim 1$, $\Delta E \gg kT$, so $\langle x \rangle \cong \frac{1.2}{\alpha^{1/3}} nm$

Can think of $x_{ox}^{eff} = x_{ox} + \langle x \rangle \frac{\epsilon_{ox}}{\epsilon_{Si}} = x_{ox} + \frac{\langle x \rangle}{3}$