

2 / Carrier Scattering

As carriers traverse a device, their motion is frequently interrupted by collisions with impurity atoms, phonons, crystal defects, or other carriers. In this chapter we examine carrier scattering and evaluate the transition rate, $S(\mathbf{p}, \mathbf{p}')$, which is the probability per unit time that a carrier with crystal momentum $\mathbf{p}$ scatters to a state with crystal momentum $\mathbf{p}'$. Our approach is based on Fermi's golden rule as described in Section 1.5. The first step is to identify the scattering potential and then to evaluate the matrix element,

$$H_{p'p} = \frac{1}{\Omega} \int_{-\infty}^{\infty} e^{-i\mathbf{p}' \cdot \mathbf{r}/\hbar} U_S(\mathbf{r}) e^{i\mathbf{p} \cdot \mathbf{r}/\hbar} d^3r \quad (2.1)$$

The matrix element is evaluated using plane wave electron wave functions. The overlap integral due to the cell periodic part of the Bloch function is unity when the constant energy surfaces are spherical and the energy varies parabolically with momentum (which we'll simply call spherical, parabolic bands).

From the matrix element we find the transition rate as

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi}{\hbar} |H_{p'p}|^2 \delta[E(\mathbf{p}') - E(\mathbf{p}) - \Delta E] \quad (2.2)$$

where ΔE is the change in energy (if any) caused by the collision. [Note that the order of $\mathbf{p}$ and $\mathbf{p}'$ in $H_{p'p}$ is, by definition, opposite to their order in $S(\mathbf{p}, \mathbf{p}')$].

We won't attempt to catalog $S(\mathbf{p}, \mathbf{p}')$ for all possible scattering mechanisms but will focus, instead, on scattering by ionized impurities, by phonons, and by other carriers because they tend to be the most important. Before we begin, we define some characteristic times that are related to the transition rate and which summarize the effects of scattering upon carriers.

2.1 RELAXATION TIMES

Consider a beam of energetic carriers injected into a semiconductor at time $t = 0$, with their momenta, $\mathbf{p}_0$, aligned along the z -axis. Figure 2.1 illustrates how collisions affect such carriers. The transition rate is the rate at which carriers scatter from a specific initial state to a specific final state. The scattering rate,

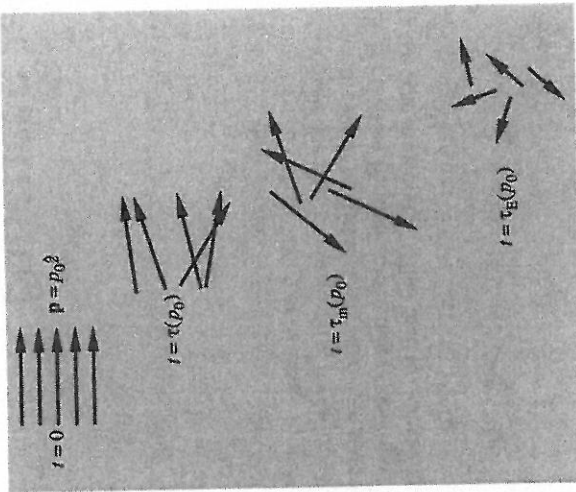


Fig. 2.1 Illustration of how collisions affect a group of electrons injected at $t = 0$ with momentum $\mathbf{p}_0$.

$$\frac{1}{\tau(\mathbf{p}_0)} = \sum_{\mathbf{p}', \uparrow} S(\mathbf{p}_0, \mathbf{p}') \quad (2.3)$$

is the rate at which carriers with momentum $\mathbf{p}_0$ scatter to any other state. Alternatively, $\tau(\mathbf{p}_0)$ is the average time between collisions (also known as the lifetime of the state, $\mathbf{p}_0$). The vertical arrow below the sum is used to indicate that the sum over final states includes only those whose spin is parallel to that of the incident carrier (the scattering mechanisms we consider do not flip the carrier's spin). A factor of $1 - f(\mathbf{p}')$, where $f(\mathbf{p}')$ is the probability that the state at $\mathbf{p}'$ is occupied, should, in general, be included in eq. (2.3). We assume a nondegenerate semiconductor, so that there is a high probability that the state at $\mathbf{p}'$ is empty.

Some important scattering mechanisms are not isotropic. Instead, they tend to deflect carriers by small angles. Such a case is illustrated in Fig. 2.1, which shows that even after $\tau(\mathbf{p}_0)$ seconds, the carriers retain a memory of their incident momentum. To evaluate the rate at which the $\hat{z}$ -directed momentum is relaxed, we need to weight each collision by the fractional change in the $\hat{z}$ -directed momentum. The result is

$$\frac{1}{\tau_m(\mathbf{p}_0)} = \sum_{\mathbf{p}', \uparrow} S(\mathbf{p}_0, \mathbf{p}') (1 - p'_z/p_{z0}) = \sum_{\mathbf{p}', \uparrow} S(\mathbf{p}_0, \mathbf{p}') (1 - \frac{p'_z}{p_0} \cos \alpha) \quad (2.4)$$

where α is the polar angle between the incident and scattered momenta as illustrated in Fig. 2.2a. $\tau_m(\mathbf{p}_0)$ is known as the *momentum relaxation time* and, as Fig. 2.1 indicates, is the time required to randomize the momentum.

We may also be interested in the time required to dissipate carrier energy, which is denoted by the lengths of the vectors in Fig. 2.1. As Fig. 2.1 indicates, it is quite possible to relax the injected momentum by elastic scattering without affecting the energy of the carriers. To find the energy relaxation rate, we weight each collision by the fractional change in energy and find

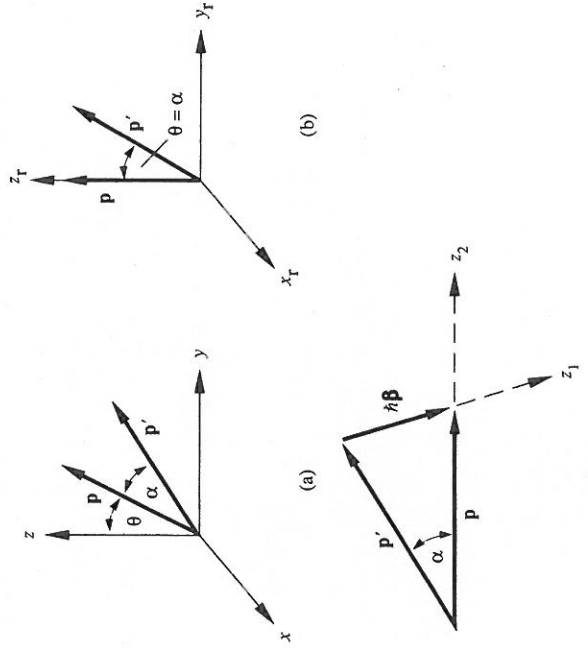


Fig. 2.2 (a) Illustration of a scattering event showing the incident and scattered momenta and the polar angle, α . (b) Illustration of the same scattering event with the coordinate axes oriented so that $\theta = \alpha$. (c) Definition of $\hbar\beta$, the momentum change resulting from scattering. We also show two choices for orienting the z -axis. The first choice, denoted by z_1 , is used to evaluate the matrix element in eq. (2.19), and the second choice, denoted by z_2 , is used for evaluating the momentum relaxation time in eq. (2.23).

$$\frac{1}{\tau_E(\mathbf{p}_0)} = \sum_{\mathbf{p}', \uparrow} S(\mathbf{p}_0, \mathbf{p}') \left(1 - \frac{E(\mathbf{p}')}{E(\mathbf{p}_0)}\right) \quad (2.5)$$

where $\tau_E(\mathbf{p}_0)$ is the *energy relaxation time*. We routinely evaluate $\tau(\mathbf{p}_0)$, $\tau_m(\mathbf{p}_0)$, and $\tau_E(\mathbf{p}_0)$ from $S(\mathbf{p}_0, \mathbf{p}')$ because knowledge of these three characteristic times concisely summarizes how collisions affect carriers.

2.1.1 Example

In Section 1.5, we evaluated $S(\mathbf{p}, \mathbf{p}')$ for two simple perturbing potentials. For the δ -function perturbing potential, we found

$$S(\mathbf{p}, \mathbf{p}') = \frac{C}{\Omega} \delta(E' - E) \quad (2.6)$$

where C is a constant and Ω a normalization volume. For this potential, the scattering rate is

$$\frac{1}{\tau(\mathbf{p})} = \frac{C}{\Omega} \sum_{\mathbf{p}', \uparrow} \delta(E' - E) \quad (2.7)$$

which is evaluated by converting the sum to an integral,

$$\frac{1}{\tau(\mathbf{p})} = \frac{C}{8\pi^2 \hbar^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi \delta\left(\frac{p'^2}{2m} - \frac{p^2}{2m}\right) p'^2 dp' \sin \theta d\theta d\phi \quad (2.8)$$

Note that in converting the sum to an integral, we used one-half of the prescription stated in eq. (1.49) because only those final states with spin parallel to the incident electron's are available.

The integral in eq. (2.8) is readily evaluated to find

$$\frac{1}{\tau(\mathbf{p})} = C \frac{(2m^*)^{3/2}}{4\pi^2 \hbar^3} E'^{1/2}(p) = \frac{C}{2} g_C(E) \quad (2.9)$$

where $g_C(E)$ is the density of states at energy E . The result states that the scattering rate for a carrier with energy E is proportional to the density of states at E . The more final states available, the higher the scattering rate.

To find the momentum relaxation time, we make use of eq. (2.4). The resulting integral is much like eq. (2.8) but with a second term as well. If we align our coordinate axes so that $\hat{z}$ is directed along the incident momentum, then $\alpha = \theta$, as illustrated in Fig. 2.2b. The second contribution to the integral then involves integration over

2.2

$\cos \theta \sin \theta d\theta$, which integrates to zero. We conclude that $\tau_m = \tau$ for this scattering potential. More generally, the momentum relaxation time is always equal to the scattering time when the transition rate is isotropic [that is, $S(\mathbf{p}, \mathbf{p}')$ contains no dependence on θ or ϕ].

Finally, we consider the energy relaxation time. It is apparent that this scattering mechanism is elastic, so $E' = E$ and from eq. (2.5) we conclude that $\tau_E = \infty$. The result simply states that elastic scattering cannot relax energy.

2.2 SCATTERING BY IONIZED IMPURITIES

Carriers are scattered when they encounter the electric field of an ionized impurity. We might assume that the scattering potential is Coulombic,

$$U_S(\mathbf{r}) = \frac{q^2}{4\pi\kappa_S\epsilon_0 r} \quad (2.10)$$

but the ionized impurity attracts mobile carriers which *screen* the potential. The electrostatic potential due to both the ionized impurity and mobile carriers is found by solving Poisson's equation in spherical coordinates,

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dV}{dr} \right] = \frac{q}{\kappa_S\epsilon_0} (n - N_D^+) \quad (2.11)$$

where an n -type semiconductor has been assumed. Space charge neutrality dictates that $n = n_0 = N_D^+$ (with a corresponding electrostatic potential of $V = V_0$) on a macroscopic scale, but on a microscopic scale, perturbations in the potential and carrier density exist. By writing the potential and carrier density as $V = V_0 + \delta V$ and $n = n_0 + \delta n$ and substituting in eq. (2.11), we find

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\delta V}{dr} \right] = \frac{q\delta n}{\kappa_S\epsilon_0} \quad (2.12)$$

To find δn , recall that

$$n = N_C e^{[E_F - E_C(\mathbf{r})]/k_B T}$$

for a nondegenerate semiconductor and that

$$E_{C0}(\mathbf{r}) = \text{constant} - qV(\mathbf{r})$$

and so small perturbations in carrier density can be related to perturbations in the potential by

$$\delta n = \frac{\partial n}{\partial V} \delta V = \frac{q n_0}{k_B T} \delta V$$

After inserting this result in Poisson's equation, we find

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\delta V}{dr} \right] = \frac{q^2 n_0}{\kappa_S \epsilon_0 k_B T} \delta V = \delta V / L_D^2 \quad (2.13)$$

where

$$L_D \equiv \sqrt{\frac{\kappa_S \epsilon_0 k_B T}{q^2 n_0}} \quad (2.14)$$

is known as the *Debye length*. [For degenerate semiconductors, Fermi-Dirac statistics apply, and the screening length, eq. (2.14), must be generalized as discussed in homework problem 2.3.]

The solution to eq. (2.13),

$$\delta V = \frac{A}{r} e^{-r/L_D} \quad (2.15)$$

shows that perturbations in potential decay exponentially with distance from the perturbation. If the perturbation is due to an ionized impurity, we should recover the Coulombic potential as $r \rightarrow 0$, so

$$A = \frac{q^2}{4\pi\kappa_S\epsilon_0}$$

For an *n*-type semiconductor in which the carrier density is approximately the ionized donor density, the appropriate perturbing potential is, therefore, the *screened Coulomb potential*,

$$U_S(\mathbf{r}) = \frac{q^2}{4\pi\kappa_S\epsilon_0 r} e^{-r/L_D} \quad (2.16)$$

On the other hand, when the mobile carrier density is low, as it is in the depletion region of a *p-n* junction, the unscreened Coulomb potential [eq. (2.10)] is used.

Having identified the perturbing potential for ionized impurity scattering, we evaluate the matrix element from eq. (2.1) as

$$H_{p'p} = \frac{1}{\Omega} \left[\frac{q^2}{4\pi\kappa_S\epsilon_0} \right] \int e^{-i\mathbf{p}' \cdot \mathbf{r}/\hbar} \frac{e^{-r/L_D}}{r} e^{i\mathbf{p} \cdot \mathbf{r}/\hbar} d^3r$$

or

$$H_{p'p} = \frac{1}{\Omega} \left[\frac{q^2}{4\pi\kappa_S\epsilon_0} \right] \int_0^{2\pi} \int_0^\pi \int_0^\infty e^{-r'/L_D} e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{r}/\hbar} r \sin\theta dr d\theta d\phi \quad (2.17)$$

According to the geometry of the scattering event (displayed in Fig. 2.2c),

$$\hbar \boldsymbol{\beta} = \mathbf{p} - \mathbf{p}' \quad (2.18a)$$

and

$$\hbar \beta = 2p \sin\left(\frac{\alpha}{2}\right) \quad (2.18b)$$

The latter follows from Fig. 2.2c because $p = p'$ for elastic scattering. With the aid of eq. (2.18a), the matrix element becomes

$$H_{p'p} = \frac{1}{\Omega} \left[\frac{q^2}{4\pi\kappa_S\epsilon_0} \right] \int_0^\infty \int_0^\pi \int_0^{2\pi} r e^{-r'/L_D} e^{i\beta r \cos\theta/\hbar} d(\cos\theta) d\phi dr \quad (2.19)$$

which can be integrated to find

$$H_{p'p} = \frac{-q^2}{\Omega\kappa_S\epsilon_0} \frac{1}{(\beta/\hbar)^2 + 1/L_D^2} \quad (2.20)$$

According to eq. (2.2), the transition rate due to scattering from a single ionized impurity is

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi}{\hbar} \left[\frac{q^2}{\Omega\kappa_S\epsilon_0} \right]^2 \frac{\delta(E' - E)}{[(\beta/\hbar)^2 + 1/L_D^2]^2}$$

Finally, we multiply by $N_i\Omega$, the number of impurities in the normalization volume, and make use of eq. (2.18b) to write

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi N_i q^4}{\hbar^2 \kappa_3^2 \epsilon_0^2 \Omega} \frac{\delta(E' - E)}{\left[4\left(\frac{p}{\hbar}\right)^2 \sin^2\left(\frac{\alpha}{2}\right) + \frac{1}{L_D^2} \right]^2} \quad (2.21a)$$

Equation (2.21a) describes scattering from an ionized impurity whose potential is screened by mobile carriers. When mobile carriers are absent, $L_D \rightarrow \infty$ and eq. (2.21a) becomes

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi N_i q^4}{\hbar^2 \kappa_3^2 \epsilon_0^2 \Omega} \frac{\delta(E' - E)}{16\left(\frac{p}{\hbar}\right)^4 \sin^4\left(\frac{\alpha}{2}\right)} \quad (2.21b)$$

Equation (2.21b) applies when mobile carriers are absent, which occurs, for example, in the depletion region of a p - n junction. When the carrier density is very high, L_D is small so that $1/L_D^2$ in the denominator of eq. (2.21a) dominates. For this strongly screened case,

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi N_i q^4 L_D^4}{\hbar^2 \kappa_3^2 \epsilon_0^2 \Omega} \delta(E' - E) \quad (2.21c)$$

and the transition rate has the form of eq. (2.6), which describes scattering from a δ -function potential.

A plot of $S(\mathbf{p}, \mathbf{p}')$ versus α (the polar angle between the incident and scattered momenta) is displayed in Fig. 2.3. The important point to note is that ionized impurity scattering tends to deflect carriers by small angles — except when it is strongly screened, in which case ionized impurity scattering is isotropic.

From the transition rate, the various relaxation times can also be evaluated. Evaluation of the scattering rate is the subject of homework problem 2.1. To find the momentum relaxation time, we evaluate

$$\begin{aligned} \frac{1}{\tau_m(p)} &= \sum_{\mathbf{p}', \uparrow} S(\mathbf{p}, \mathbf{p}') \left(1 - \frac{p'}{p} \cos \alpha\right) \\ &= \frac{\Omega}{8\pi^3 \hbar^3} \int_0^{2\pi} \int_0^\pi S(\mathbf{p}, \mathbf{p}') \left(1 - \frac{p'}{p} \cos \alpha\right) p'^2 dp' d(\cos \theta) d\phi \end{aligned} \quad (2.22)$$

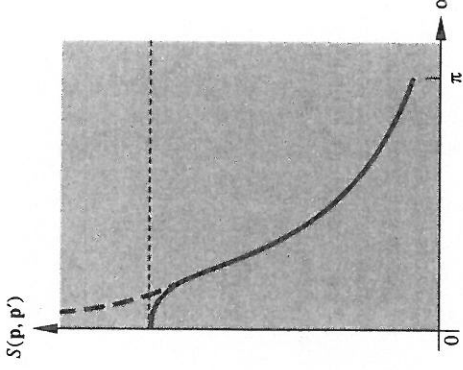


Fig. 2.3 $S(\mathbf{p}, \mathbf{p}')$ versus α for ionized impurity scattering; the solid line is for a screened Coulomb potential, the dashed line for the unscreened ($L_D \rightarrow \infty$) Coulomb potential, and the dotted line for the strongly screened ($L_D \rightarrow \infty$) Coulomb potential.

If we orient the z -axis so that it points along the initial momentum (see Fig. 2.2c), then $\alpha = \theta$, $\mathbf{p} = p\hat{z}$, $\mathbf{p}' = p' \cos \theta$, and eq. (2.22) becomes

$$\frac{1}{\tau_m(p)} = \frac{N_i q^4}{2\pi \kappa_3^2 \epsilon_0^2 \hbar^4} \int_0^\pi \int_{-1}^1 \frac{(1 - \cos \theta) d(\cos \theta)}{\left[4\left(\frac{p}{\hbar}\right)^2 \sin^2\left(\frac{\theta}{2}\right) + \frac{1}{L_D^2} \right]^2} \delta(E' - E) p'^2 dp' \quad (2.23)$$

(We also made use of the fact that $p = p'$ because ionized impurity scattering is elastic.) With the substitution $x = 1 - \cos \theta$ the first integral in eq. (2.23) becomes

$$I_\theta = \int_0^2 \frac{x dx}{\left[2\left(\frac{p}{\hbar}\right)^2 x + \frac{1}{L_D^2} \right]^2} = \frac{1}{4\left(\frac{p}{\hbar}\right)^4} \left[\ln(1 + \gamma^2) - \frac{\gamma^2}{1 + \gamma^2} \right]$$

where

$$\gamma^2 \equiv 4L_D^2(p/\hbar)^2 = 8m^* E(p) L_D^2 / \hbar^2 \quad (2.24)$$

After inserting this result for f_0 in eq. (2.23) and integrating over p , we find

$$\tau_m(p) = \frac{16\sqrt{2m^*}\pi\kappa_3^2\epsilon_0^2}{N_I q^4} \left[\ln(1 + \gamma^2) - \frac{\gamma^2}{1 + \gamma^2} \right]^{-1} E^{3/2}(p) \quad (2.25)$$

Figure 2.4, a plot of $1/\tau_m(p)$ versus energy, shows that carriers with high kinetic energy have long momentum relaxation times. The increase in τ_m with energy occurs because rapidly moving carriers are deflected less by ionized impurities. As a consequence, the influence of ionized impurity scattering decreases at high temperatures or at high electric fields because both increase the carriers' kinetic energy. Equation (2.25), which assumes a screened Coulomb potential, is a simple version of a theory formulated by Brooks and Herring.

2.2.1 Unscreened Coulomb Scattering

When the free carrier density is low, screening is minimal, and the transition rate is given by eq. (2.21b). When we attempt to use eq. (2.21b), however, we are confronted with the unpleasant fact that it predicts that the probability of scattering approaches infinity as the angle of deflection approaches zero. When the carrier is far from an

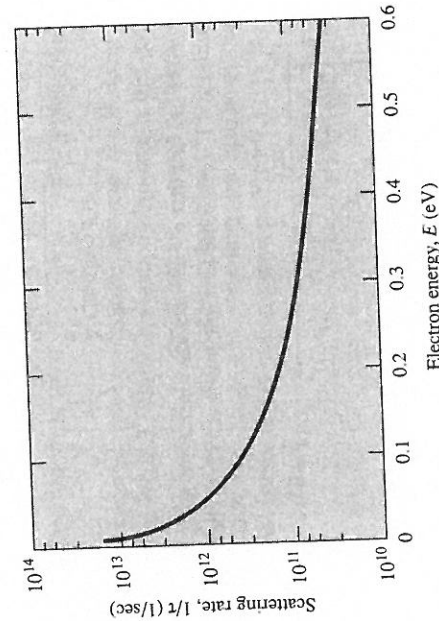


Fig. 2.4 Ionized impurity momentum scattering rate versus energy for electrons in GaAs ($N_I = 10^{17} \text{ cm}^{-3}$, $T = 300^\circ \text{ K}$).

ionized impurity, as measured by the *impact parameter* defined in Fig. 2.5, it is deflected very little. But if the carrier's impact parameter is greater than

$$b_{\max} = \frac{1}{2} N_I^{-1/3} \quad (2.26)$$

which is one-half the spacing between impurities, then the scattering simply occurs from a neighboring impurity. The result is that a minimum scattering angle exists, so the infinity in $S(\mathbf{p}, \mathbf{p}')$ is never approached. Since the impact parameter and the angle of deflection are related by Rutherford's theory as

$$b = \frac{q^2}{8\pi\kappa_3\epsilon_0 E(p)} \cot(\alpha/2) \quad (2.27)$$

the minimum deflection angle is determined from

$$\frac{1}{\sin^2(\theta_{\min}/2)} = 1 + \gamma_{CW}^2 \quad (2.28)$$

where

$$\gamma_{CW} = \frac{4\pi\kappa_3\epsilon_0 E(p)}{q^2 N_I^{1/3}} = \frac{b_{\max}}{[q^2/8\pi\kappa_3\epsilon_0 E(p)]} \quad (2.29)$$

The subscript CW is for Conwell and Weisskopf, who first performed the calculation about to be described.

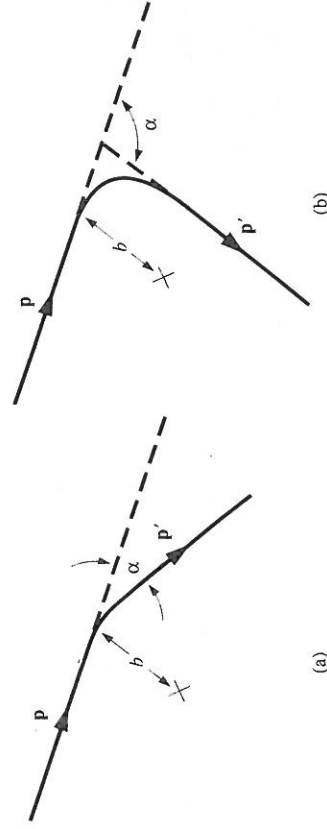


Fig. 2.5 Effect of impact parameter on ionized impurity scattering. (a) Large impact parameter; (b) small impact parameter.

The momentum scattering rate in the Conwell-Weisskopf approach is obtained from eq. (2.23) by letting $L_D \rightarrow \infty$. If we orient the coordinate system so that $\alpha = \theta$ and integrate from $\theta_{\min}$ to π , we find

$$\frac{1}{\tau_m(p)} = \frac{N_I q^4}{2\pi \kappa_3^2 \epsilon_0^2 \hbar^4} \int_0^{\infty \cos \theta_{\min}} \int_{-1}^1 \frac{(1 - \cos \theta) d(\cos \theta)}{16(p/\hbar)^4 \sin^4(\theta/2)} \delta(E' - E) p'^2 dp' \quad (2.30)$$

Equation (2.30) works out much like the integrals in the Brooks-Herring approach with the result that

$$\tau_m(p) = \frac{16\pi \sqrt{2m^*} \kappa_3^2 \epsilon_0^2}{N_I q^4} \left[\frac{1}{\ln(1 + \gamma_{CW}^2)} \right] E^{3/2}(p) \quad (2.31)$$

Except for the term in brackets, eq. (2.31) is much like the Brooks-Herring result.

2.2.2 Strongly Screened Coulomb Scattering

When the mobile carrier density is very high, the Coulomb potential is strongly screened, and the transition rate is given by eq. (2.21c). Because the result has the simple form due to a δ -function scattering potential, the results of the example considered in Section 2.1 apply, and we find

$$\frac{1}{\tau_m(p)} = \frac{1}{\tau(p)} = \frac{\pi N_I q^4 L_D^4}{\hbar \kappa_3^2 \epsilon_0^2} g_C(E) \quad (2.32)$$

2.2.3 Discussion

We have outlined three simple treatments of ionized impurity scattering: the first assumed that free carriers were present to screen the impurity potential, the second assumed that free carrier screening was very weak, and the third assumed that free carrier screening was very strong. The final results [eqs. (2.25), (2.31), and (2.32)] can be written in the form

$$\tau_m(p) = \tau_0 (E/k_B T)^s \quad (2.33)$$

where "s" is a characteristic exponent. Because the terms in brackets in eqs. (2.25) and (2.31) vary slowly with energy, τ_0 is approximately constant. When the screening is moderate or weak, $s = 3/2$, but when it is very strong, $s = -1/2$.

Finally, we should mention that the several theoretical difficulties we've glossed over actually make ionized impurity scattering rather difficult to treat accurately. The Born approximation, on which the golden rule is based, fails for slow moving electrons in a Coulomb potential. Moreover, our simple expedient of computing the scattering due to a single impurity and then multiplying by the number of impurities does not account for the interference effects that occur as the electron wave propagates through the array of randomly placed scatterers. The expressions developed in this section comprise the standard approach for treating ionized impurity scattering in semiconductors; they serve reasonably well for many applications. The limitations of this approach are described in ref. [2.10].

2.3 ENERGY AND MOMENTUM CONSERVATION IN PHONON SCATTERING

When a carrier collides with a phonon, both energy and momentum are conserved. Energy and momentum conservation impose constraints on the maximum wave vector change and, therefore, on which phonons may participate in scattering events. Our purpose in this section is to demonstrate that for *intravalley* scattering, in which the carrier resides in the same valley before and after scattering, the phonons involved are those with wave vectors near the center of the Brillouin zone. This result is important because it means that we can employ the simple dispersion relations sketched in Fig. 1.13b.

Conservation of energy states that

$$E(\mathbf{p}') = E(\mathbf{p}) \pm \hbar \omega(\boldsymbol{\beta}) \quad (2.34)$$

where $\hbar \omega(\boldsymbol{\beta})$ is the energy of the phonon and $\pm$ denotes scattering by phonon absorption or emission. For spherical, parabolic energy bands, eq. (2.34) becomes

$$\frac{p'^2}{2m^*} = \frac{p^2}{2m^*} \pm \hbar \omega(\boldsymbol{\beta}) \quad (2.35)$$

Momentum conservation also applies and states that

$$\mathbf{p}' = \mathbf{p} \pm \hbar \boldsymbol{\beta} \quad (2.36)$$

where $\hbar \boldsymbol{\beta}$ is the momentum of the phonon. The dot product of eq. (2.36) with itself can be inserted in eq. (2.35) and rearranged to arrive at

$$\hbar\beta = 2p \left[\mp \cos \theta \pm \frac{\omega}{v(p)\beta} \right] \quad (2.37)$$

where θ is the polar angle between $\mathbf{p}$ and β . Equation (2.37) simultaneously states energy and momentum conservation. Since $\cos \theta$ is restricted to the range between -1 and $+1$, eq. (2.37) determines the minimum and maximum values for the wave vectors of the phonons involved in intravalley carrier scattering.

2.3.1 Intravalley Acoustic Phonon Scattering

For acoustic phonons with wave vectors near the zone center, $\omega/\beta = v_s$ (the sound velocity) and eq. (2.37) becomes

$$\hbar\beta = 2p \left[\mp \cos \theta \pm \frac{v_s}{v(p)} \right] \quad (2.38)$$

The maximum wave vector occurs when $\theta = \pi$ for phonon absorption and when $\theta = 0$ for phonon emission, so we conclude that

$$\hbar\beta_{\max}(\text{AP}) = 2p \left[1 \pm \frac{v_s}{v(p)} \right] \quad (2.39)$$

Since $v_s \approx 10^5$ cm/sec and $v(p) \approx 10^7$ cm/sec for a thermal average carrier,

$$\hbar\beta_{\max}(\text{AP}) \approx 2p \quad (2.40)$$

for a typical acoustic phonon (AP) scattering event. As shown in Fig. 2.6, eq. (2.40) is what we expect for elastic scattering, which suggests that acoustic phonon scattering is nearly elastic. Note also that eq. (2.39) and the requirement that the wave vector be positive dictate that a carrier can't scatter by emitting an acoustic phonon unless its velocity exceeds the sound velocity.

To gauge the magnitude of $\beta_{\max}$, it should be compared with π/a , the approximate half-width of the Brillouin zone. From eq. (2.40) we find that



Fig. 2.6 Illustration of maximum phonon wave vector for elastic scattering.

$$\frac{\beta_{\max}}{\pi/a} = \frac{2m^* v(p)}{\hbar \pi/a} \approx 1/4$$

[for $m^* \approx m_0$, $v(p) \approx 10^7$ cm/sec, and $a = 5 \text{ \AA}$]. We conclude that acoustic phonon scattering involves phonons with wave vectors near the center of the Brillouin zone.

The maximum change in carrier energy resulting from acoustic phonon scattering is readily estimated from

$$\Delta E_{\max} = \hbar \omega_{\max} = \hbar \beta_{\max} v_s \approx 10^{-3} \text{ eV}$$

Acoustic phonon scattering is often taken to be elastic because ΔE is small compared with $k_B T_L$, except at very low temperatures.

2.3.2 Intravalley Optical Phonon Scattering

For optical phonons, ω is nearly constant at $\omega(\beta) \approx \omega_0$. According to Table 2.1 at the end of this chapter, the optical phonon energy, $\hbar \omega_0$, is typically tens of millielectronvolts, which is comparable to the thermal energy of an average carrier at room temperature. As a consequence, optical phonon scattering can rarely be considered elastic. With a constant $\omega(\beta)$, eq. (2.37) produces a quadratic equation for β that can be solved to obtain

$$\hbar\beta = p \left[-\cos \theta + \sqrt{\mp \cos^2 \theta \pm \frac{\hbar \omega_0}{E(p)}} \right] \quad (2.41)$$

from which we obtain the maximum wave vector of the optical phonon (OP) as

$$\hbar\beta_{\max}(\text{OP}) = p \left[1 + \sqrt{1 \pm \frac{\hbar \omega_0}{E(p)}} \right] \quad (2.42)$$

Because the maximum wave vector must be a positive number, eq. (2.42) states that carriers cannot scatter by emitting optical phonons unless their energy exceeds the optical phonon energy. When typical numbers are inserted in eq. (2.42), we find that only optical phonons near the center of the Brillouin zone participate in intravalley scattering.

Application of energy and momentum conservation has identified the phonons involved in intravalley scattering as those whose wave vectors are near the center of the Brillouin zone. Acoustic phonon scattering is usually considered to be elastic, but optical phonon scattering can be considered elastic only for high energy carriers. Figures 2.7 and 2.8 illustrate typical carrier wave vectors before and after scattering from acoustic and optical phonons.

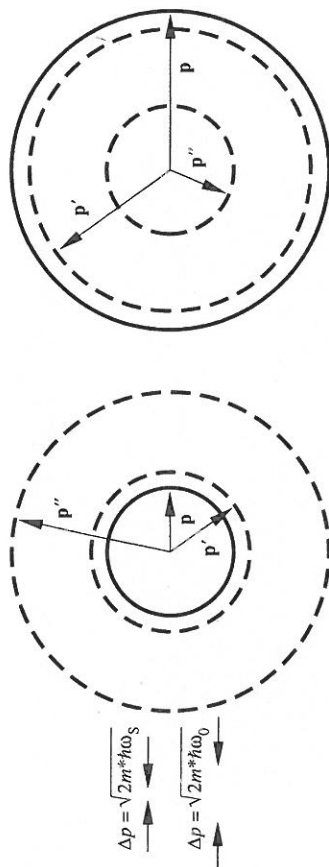


Fig. 2.7 Illustration of carrier scattering by absorption of an acoustic or optical phonon. The incident momentum is p . The momentum after scattering by absorption of an acoustic phonon is p' , and the momentum after scattering by absorption of an optical phonon is p'' . After Ridley (Ref. [2.1]).

Fig. 2.8 Illustration of carrier scattering by emission of an acoustic or optical phonon. The incident momentum is p . The momentum after scattering by emission of an acoustic phonon is p' , and the momentum after scattering by emission of an optical phonon is p'' . After Ridley (Ref. [2.1]).

2.4 THE ELECTRON-PHONON INTERACTION

Because a semiconductor's band structure is determined by the crystal potential, it is influenced by changes in lattice spacing. A semiconductor under pressure has a perturbed lattice constant and band structure as sketched in Fig. 2.9a. For a small change in lattice constant, we expect that

$$\delta E_C = D_C \frac{\delta a}{a} \quad (2.43a)$$

and that

$$\delta E_V = D_V \frac{\delta a}{a} \quad (2.43b)$$

where D_C and D_V , the *deformation potentials*, can be deduced from experiments and have been characterized for common semiconductors (refer to Table 2.1 at the end of this chapter). The change in effective mass with lattice spacing is small and is neglected. Lattice vibrations deform the lattice, producing a "grating" in the band

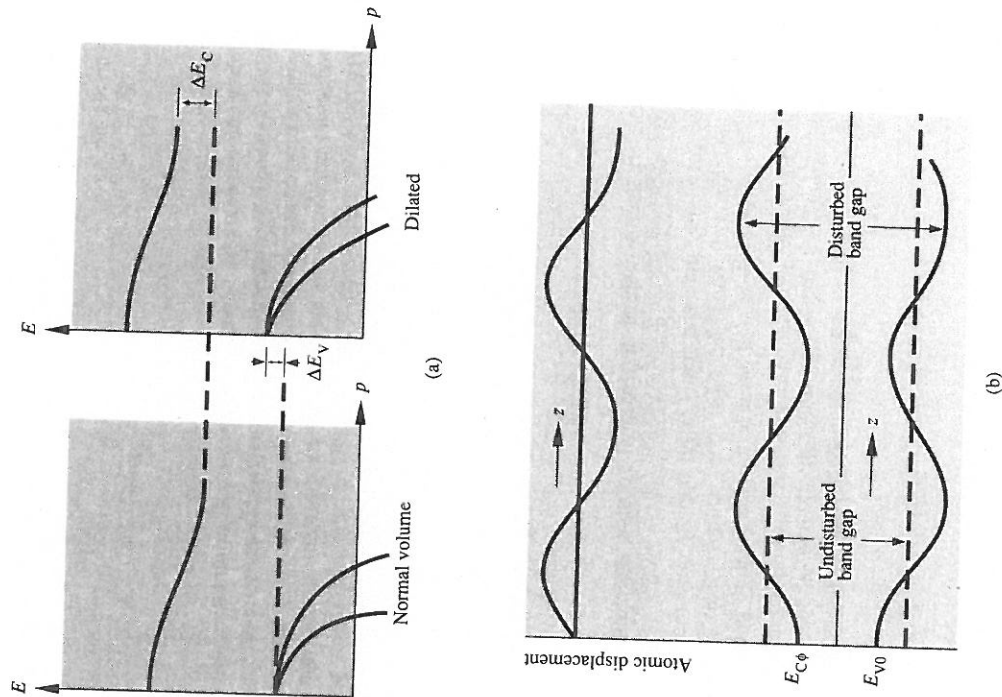


Fig. 2.9 (a) Effect of a change in lattice spacing on the band structure of a semiconductor. After Harrison, *Solid State Theory*, Dover, New York, 1980. (b) Band edge variation produced by a lattice vibration. After Nag, *Electron Transport in Compound Semiconductors*, Springer-Verlag, New York, 1980.

edges as sketched in Fig. 2.9b. Electrons and phonons interact when carrier waves scatter off of this grating.

Consider an elastic wave,

$$u(x, t) = A \beta e^{i(\pm \beta x - \omega t)} \quad (2.44)$$

propagating in a one-dimensional lattice. The use of a continuum description in eq. (2.44) is justified because only long wavelength (small β) phonons are effective in intravalley scattering. For acoustic phonons, which displace neighboring atoms in the same direction, changes in lattice spacing are produced by the strain $\partial u / \partial x$, not by the displacement $u(x, t)$. Motivated by eq. (2.43a), we write the interaction potential for acoustic phonons as

$$U_{AP}(x, t) = D_A \frac{\partial u}{\partial x} \quad (2.45)$$

To extend these arguments to three-dimensional crystals, the phonon's polarization must be considered. Since transverse elastic waves produce no first order change in the lattice spacing, the perturbing potential [eq. (2.45)] applies only to longitudinal phonons. For semiconductors with band structures more complex than a simple spherical band, the problem is further complicated because the deformation potential becomes a tensor (Refs. [2.1] and [2.4]).

For optical phonons, which displace neighboring atoms in opposite directions, the displacement produces a change in lattice spacing directly, so the scattering potential for optical phonons is written as

$$U_{OP}(x, t) = D_O u(x, t) \quad (2.46)$$

where D_O is the optical deformation potential. In real, three-dimensional semiconductors, optical phonon vibrations consist of one sublattice moving against the other. In contrast to the simple change in volume of the unit cell produced by longitudinal acoustic phonons, optical phonon scattering is sensitive to the symmetry of the crystal. Selection rules forbid optical phonon scattering of electrons at $\mathbf{p} = (0, 0, 0)$ and along $\langle 100 \rangle$ directions (which includes electrons in the conduction bands of GaAs and Si).

2.3

Optical phonon scattering does occur for holes, and for conduction band electrons in germanium.

Acoustic and optical deformation potential scattering also occur in compound semiconductors, but an additional, very strong interaction due to the polar nature of the bonds usually dominates. In compound semiconductors such as GaAs, the bond between adjacent atoms is partially ionic; the arsenic atom acquires a slight positive charge, the gallium atom a small negative charge. The magnitude of this charge, termed the effective charge, q^* , is a fraction of the electronic charge, q , and is determined by the degree of ionicity of the bond (Ref. [2.1]). Deformation of the lattice by phonons perturbs the dipole moment between atoms, which results in an electric field that scatters carriers. Polar scattering may be due to either acoustic or optical phonons. Polar acoustic phonon scattering, known as *piezoelectric scattering*, can be important at very low temperatures in very pure semiconductors. Polar optical phonon scattering is very strong and is typically the dominant scattering mechanism in GaAs near room temperature.

For longitudinal optical phonons, the displacement perturbs the dipole moment directly according to

$$\delta p = q^* u \quad (2.47)$$

The electric field due to the perturbed dipole moment is obtained from the relation

$$D = \epsilon_0 \epsilon + \delta p \quad (2.48)$$

and from the assumption of zero macroscopic, free charge (that is, $\nabla \cdot D = 0$). [In eq. (2.48), δp is the change in the dipole moment per unit volume produced by the elastic wave.] Since the fields and polarization are due to elastic waves, they vary as

$$D = D_x e^{i\beta x}$$

and so

$$\nabla \cdot D = \frac{\partial D}{\partial x} = i\beta D = 0$$

implies that $D = 0$. From eq. (2.48) with $D = 0$, we find

$$\epsilon = -\frac{\delta p}{\epsilon_0} \quad (2.49)$$

where

$$\delta P = \frac{\delta p}{V_u} \quad (2.50)$$

with V_u being the volume of a unit cell. From eq. (2.49), we find the electric field as

$$\epsilon = -\frac{q u}{\epsilon_0 V_u} \quad (2.51)$$

which is integrated to find the interaction potential,

$$U_S(x, t) = -q \int \epsilon dx \quad (2.52)$$

For polar optical phonon scattering, the resulting perturbing potential is

$$U_{POP} = \frac{q q^* u}{i \beta V_u \epsilon_0} \quad (2.53)$$

It is common to measure the strength of the polar interaction by the low and high frequency dielectric constants, which are easily measured, rather than by the effective charge, q^* , on the dipole. The two approaches are related by (Ref. [2.1])

$$\left[\frac{q^*}{V_u} \right]^2 = \frac{\epsilon_0 \rho \omega_0^2}{\kappa_0} \left[\frac{\kappa_0}{\kappa_\infty} - 1 \right] \quad (2.54)$$

which is understood as follows. At low frequencies, the dielectric constant includes a contribution due to the dipole between atoms, but at high frequencies this dipole cannot respond to the signal so that $\kappa_\infty < \kappa_0$. The factor, $\kappa_0/\kappa_\infty - 1$, is a measure of the strength of the dipole.

In the following two sections, the transition rates, $S(\mathbf{p}, \mathbf{p}')$, and the various relaxation times of interest are computed from the interaction potentials identified in this section. We first consider acoustic and optical deformation potential scattering, which occur in both polar and nonpolar semiconductors, and then treat polar optical phonon scattering, which is often the dominant mechanism in semiconductors such as GaAs.

2.5 DEFORMATION POTENTIAL SCATTERING

To evaluate scattering rates for acoustic phonon scattering via the deformation potential (ADP scattering), we begin with the golden rule [eq. (2.2)] and set $\Delta E = \pm \hbar \omega_s$, where $\pm$ refers to a phonon absorption and emission, respectively. For a lattice vibration

described by eq. (2.44), the scattering potential for acoustic deformation potential (ADP) scattering [eq. (2.45)] becomes

$$U_{AP} = \pm i \beta D_A u \quad (2.55)$$

which is a traveling wave potential much like the one we considered in the second example discussed in Section 1.5. By analogy with eq. (1.106), we find

$$|H_{p'p}|^2 = (D_A \beta A_\beta)^2 \delta(\mathbf{p}' - \mathbf{p} \mp \hbar \beta) \quad (2.56)$$

where the δ -function simply states that momentum is conserved.

Although we have been treating phonons as classical waves with an amplitude A_β , they are really harmonic oscillators whose energies are quantized according to

$$E(\beta) = (N_{\omega_s} + \frac{1}{2}) \hbar \omega_s \quad (2.57)$$

where N_{ω_s} is the number of acoustic phonons as given by the Bose-Einstein factor [eq. (1.113)]. Classically, the energy is related to the square of the wave amplitude. The kinetic energy is $\frac{1}{2} M |du/dt|^2$, where $M = \rho \Omega$ is the mass in the normalization volume. The energy oscillates between kinetic and potential energy, but the total energy is constant and equal to the maximum kinetic energy. The classical total energy is

$$E(\beta) = 2\rho\omega_s^2 A_\beta^2 \Omega \quad (2.58)$$

By equating the quantum and classical results [eqs. (2.57) and (2.58)] we find

$$A_\beta^2 = \frac{(N_{\omega_s} + \frac{1}{2}) \hbar}{2\rho\omega_s\Omega}$$

which is almost, but not quite, correct.

A proper, quantum treatment of lattice vibrations shows that when a phonon is absorbed, the matrix element is proportional to the number of phonons, N_{ω_s} , but when a phonon is emitted, it is proportional to $N_{\omega_s} + 1$. The first term accounts for *stimulated emission* and the second for *spontaneous emission* (a phonon can be emitted even if no phonons are present). We conclude that to properly account for the quantum mechanical nature of phonons, the wave amplitude in eq. (2.56) should be replaced according to the prescription

$$A_{\beta}^2 \rightarrow (N_{\omega_s} + 1/2 \mp 1/2) \frac{\hbar}{2\rho\omega_s\Omega} \quad (2.59)$$

(For a proper derivation of this result, consult Volume VIII of the Modular Series on Solid State Devices, *Quantum Phenomena*, by Supriyo Datta.)

After inserting the quantum mechanics in eq. (2.56) according to the prescription [eq. (2.59)] and using the golden rule, we find

$$S(\mathbf{p}, \mathbf{p}') = \frac{\pi D_A^2 \beta^2}{\rho\omega_s\Omega} (N_{\omega_s} + 1/2 \mp 1/2) \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E \mp \hbar\omega_s) \quad (2.60)$$

which is the transition rate for acoustic phonon scattering. For typical, room temperature conditions, eq. (2.60) can be simplified considerably. We argued in Section 2.3 that under such conditions, acoustic phonon scattering can be considered as elastic, so $\hbar\omega_s$ can be set to zero. Moreover, the number of room temperature acoustic phonons is large, so $N_{\omega_s} \approx N_{\omega_s} + 1$. Finally, recalling eq. (1.114), we can invoke equipartition, $N_{\omega_s} \approx k_B T_L / \hbar\omega_s$, because $\hbar\omega_s \ll k_B T_L$. With these approximations, we simplify eq. (2.60) for room temperature applications as

$$S(\mathbf{p}, \mathbf{p}') \approx \frac{2\pi D_A^2}{\Omega c_l} \frac{k_B T_L}{\hbar} \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E) \quad (2.61)$$

where we have used

$$\frac{\omega_s}{\beta} = v_s = \sqrt{\frac{c_l}{\rho}} \quad (2.62)$$

to relate the sound velocity v_s , elastic constant c_l , and mass density ρ . Equation (2.61) is the sum of transitions due to acoustic phonon absorption and emission. Note also that for any phonon that satisfies momentum conservation, the probability of scattering is the same, which means that ADP scattering (in the elastic approximation) is isotropic.

Because ADP scattering is isotropic, the scattering and momentum relaxation rates are equal and are evaluated from

$$\begin{aligned} \frac{1}{\tau(p)} &= \frac{1}{\tau_m(p)} = \sum_{\mathbf{p}', \boldsymbol{\beta}} S(\mathbf{p}, \mathbf{p}') \\ &= \frac{2\pi D_A^2 k_B T_L}{\hbar c_l} \frac{1}{\Omega} \sum_{\mathbf{p}', \boldsymbol{\beta}} \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E) \end{aligned} \quad (2.63)$$

The two δ -functions in eq. (2.63) express conservation of momentum and energy. To evaluate the sum, we write it in terms of a single δ -function that simultaneously expresses momentum and energy conservation. Proceeding as in Section 2.3, we find

$$\delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E) = \frac{1}{\hbar v(p)} \beta [\pm \cos \theta + \hbar\beta/2m^* v(p)] \quad (2.64)$$

where θ is the angle between $\mathbf{p}$ and $\hbar\boldsymbol{\beta}$. Equation (2.64) can be used in eq. (2.63) to evaluate the scattering rate, but we find it more convenient to sum over $\boldsymbol{\beta}$ rather than $\mathbf{p}'$. (Since the two are related by momentum conservation, summing over $\boldsymbol{\beta}$ and $\mathbf{p}'$ is equivalent.) The result is

$$\frac{1}{\tau(p)} = \frac{1}{\tau_m(p)} = \frac{2\pi D_A^2 k_B T_L}{\hbar^2 c_l v(p)} \frac{1}{\Omega} \sum_{\boldsymbol{\beta}} \frac{1}{\beta} \delta[\pm \cos \theta + \hbar\beta/2m^* v(p)] \quad (2.65)$$

If we call the sum I_{β} and convert it to an integral, we find

$$I_{\beta} = \frac{1}{8\pi} \int_0^{2\pi} \int_{-1}^1 \delta[\pm \cos \theta + \hbar\beta/2m^* v(p)] d\phi d(\cos \theta) \beta d\beta$$

which we integrate first over ϕ and then over $\cos \theta$. When we integrate over $\cos \theta$, we get a contribution *only* if the argument of the δ -function goes to zero for $\cos \theta$ between -1 and $+1$, which can occur only if β is not too big and not too small. The result is that the δ -function restricts the allowed range of β from 0 to ∞ to from $\beta_{\min}$ to $\beta_{\max}$, which includes only those phonons that satisfy both energy and momentum conservation. The result is

$$I_{\beta} = \frac{1}{4\pi^2} \int_{\beta_{\min}}^{\beta_{\max}} \beta d\beta = \frac{1}{8\pi^2} (\beta_{\max}^2 - \beta_{\min}^2)$$

The maximum phonon wave vector is found by setting $\cos \theta = -1$ in the argument of the δ -function and the minimum by setting $\cos \theta = +1$. We find that $\beta_{\max} = 2m^* v(p)$ and $\beta_{\min} = 0$, which can be inserted in the expression for I_{β} and used in eq. (2.65) to find

$$\frac{1}{\tau(p)} = \frac{1}{\tau_m(p)} = \frac{\pi D_A^2 k_B T_L}{\hbar c_l} g_C(E) \quad (2.66)$$

where $g_C(E)$ is the density of states defined in eq. (2.9). The result, which shows that the scattering rate is proportional to the number of final states available, could have been obtained by simpler methods, but the procedure employed here will prove useful for more complex scattering potentials. It is worth noting and remembering that $\tau_m(p)$ for ADP scattering has the power law form of eq. (2.33) with $s = -1/2$. Figure 2.10a is a plot of the ADP scattering rate versus energy for electrons in silicon.

2.5.1 Optical Deformation Potential Scattering

The treatment of optical deformation potential (ODP) scattering proceeds much like ADP scattering, except that it can't be considered as elastic unless the carrier energy is very high. The matrix element for ODP scattering is readily found from the interaction potential [eq. (2.46)] as

$$|H_{p'p}|^2 = \frac{D_0^2 \hbar}{2\rho\omega_0\Omega} (N_0 + 1/2 \mp 1/2) \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \quad (2.67)$$

where $\hbar\omega_0$ is the optical phonon energy.

The transition rate for ODP scattering is obtained from the golden rule as

$$S(\mathbf{p}, \mathbf{p}') = \frac{\pi D_0^2}{\rho\omega_0\Omega} (N_0 + 1/2 \mp 1/2) \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E \mp \hbar\omega_0) \quad (2.68)$$

which shows that optical phonon scattering is isotropic but not elastic. Because the optical phonon energy is approximately constant with β , the ODP scattering rate is evaluated much as it was for ADP scattering to find

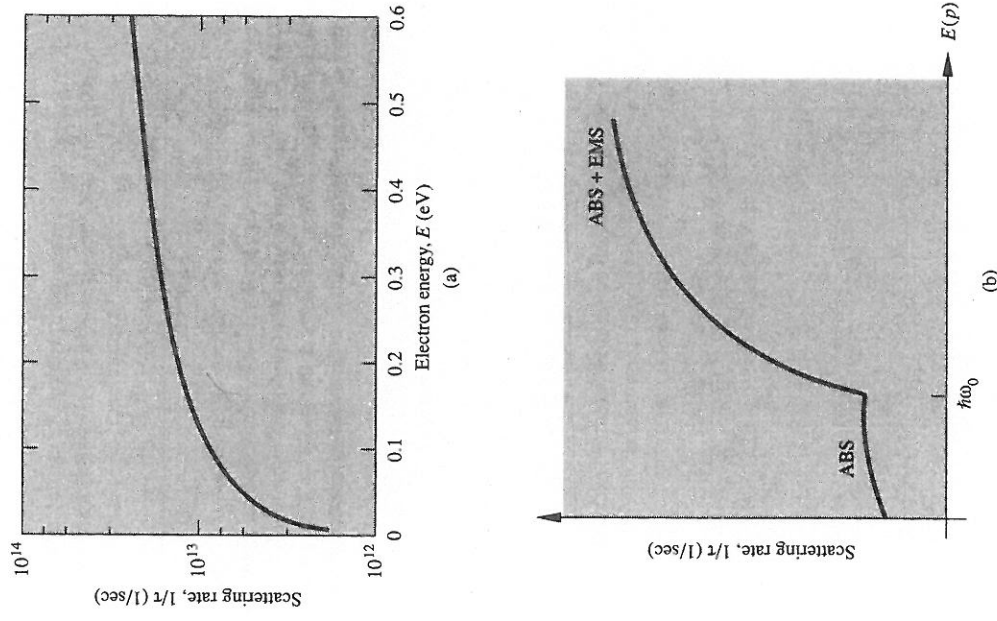


Fig. 2.10 (a) Scattering rates versus energy due to acoustic phonon scattering of electrons in silicon at 300° K. This curve was computed from eq. (2.66) using the density of states effective mass for Si and the deformation potential listed in Table 2.1. (b) Scattering rate versus carrier energy for optical phonon scattering. ABS denotes scattering by optical phonon absorption, and EMS denotes scattering by optical phonon emission.

$$\frac{1}{\tau(p)} = \frac{1}{\tau_m(p)} = \frac{\pi D_0^2}{2p\omega_0} (N_0 + \frac{1}{2} \mp \frac{1}{2}) g_c(E \pm \hbar\omega_0) \quad (2.69)$$

Note that the density of states is nonzero for positive arguments only. Equation (2.69) states that a carrier with any energy can scatter by absorbing optical phonons, but only those whose energy exceeds $\hbar\omega_0$ can emit optical phonons. The result is that a plot of the scattering rate versus carrier energy displays a threshold at $\hbar\omega_0$ above which the scattering rate greatly increases. Figure 2.10b is a sketch illustrating how the ODP scattering rate varies with energy.

High energy carriers may shed their energy by emitting optical phonons, and the energy loss can be characterized by the energy relaxation time,

$$\frac{1}{\tau_E(p)} = \sum_{p', \uparrow} S(p, p') \left(1 - \frac{E'}{E}\right) \quad (2.70)$$

For very high energy carriers, phonon emission greatly exceeds absorption, so $E(p') = E(p) - \hbar\omega_0$, and we find the energy relaxation rate as

$$\frac{1}{\tau_E(p)} = \sum_{p', \uparrow} S(p, p') \frac{\hbar\omega_0}{E(p)} = \frac{\hbar\omega_0}{E(p)} \frac{1}{\tau(p)}$$

or

$$\tau_E(p) = \left[\frac{E(p)}{\hbar\omega_0} \right] \tau(p) \quad (2.71)$$

The fraction $E(p)/\hbar\omega_0$ simply indicates how many optical phonons must be emitted to remove the carrier's kinetic energy, $E(p)$. Accordingly, the energy relaxation time may greatly exceed the momentum relaxation time.

2.6 POLAR OPTICAL PHONON SCATTERING

Phonon scattering in polar semiconductors may occur from either acoustic or optical phonons. Polar acoustic phonon, or *piezoelectric*, scattering can be important at very

low temperatures in very pure semiconductors. Scattering rates due to the piezoelectric interaction are the subject of homework problems 2.4 and 2.5. Polar optical phonon (POP) scattering, by contrast, is a very strong scattering mechanism for compound semiconductors such as GaAs. It is neither elastic nor isotropic.

The matrix element for POP scattering is found from the interaction potential [eq. (2.53)] as

$$|H_{pp'}|^2 = \frac{q^2 \omega_0 \hbar}{2\kappa_0 \epsilon_0 \beta^2 \Omega} \left(\frac{\kappa_0}{\kappa_\infty} - 1 \right) (N_0 + \frac{1}{2} \mp \frac{1}{2}) \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \quad (2.72)$$

which can be used in the golden rule to obtain

$$S(\mathbf{p}, \mathbf{p}') = \frac{\pi q^2 \omega_0}{\kappa_0 \epsilon_0 \beta^2 \Omega} \left(\frac{\kappa_0}{\kappa_\infty} - 1 \right) (N_0 + \frac{1}{2} \mp \frac{1}{2}) \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E \mp \hbar\omega_0)$$

for the transition rate due to POP scattering.

To compute the scattering rate due to POP scattering, the sum,

$$\begin{aligned} \frac{1}{\tau(p)} &= \frac{\pi q^2 \omega_0}{\kappa_0 \epsilon_0 \Omega} \left(\frac{\kappa_0}{\kappa_\infty} - 1 \right) (N_0 + \frac{1}{2} \mp \frac{1}{2}) \\ &\times \sum_{p', \uparrow} \frac{1}{\beta^2} \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E \mp \hbar\omega_0) \end{aligned} \quad (2.74)$$

must be evaluated. Proceeding as we did for ADP scattering, we first write the two δ -functions as a single one that expresses both energy and momentum conservation. Using arguments like those in Section 2.3, we find

$$\delta(\mathbf{p}' - \mathbf{p} \mp \hbar\boldsymbol{\beta}) \delta(E' - E \mp \hbar\omega_0) = \frac{m^*}{\hbar p \beta} \delta(\pm \cos \theta \mp \frac{\omega_0}{v\beta} + \frac{\hbar\beta}{2p}) \quad (2.75)$$

Let's call the sum in eq. (2.74) I_β and convert it to an integral to find

$$I_\beta = \frac{\Omega m^*}{8\pi^3 \hbar p} \int_0^\infty \int_0^{2\pi} \int_0^\pi \delta(\pm \cos \theta \mp \frac{\omega_0}{v\beta} + \frac{\hbar\beta}{2p}) \frac{d\beta}{\beta} d(-\cos \theta) d\phi$$

Note that we have again found it convenient to sum over β rather than over $\mathbf{p}'$. The integral over $\cos \theta$ is nonzero only when the argument of the δ -function goes to zero somewhere between the limits of integration. Because $\cos \theta$ is limited to the range between ± 1 , we find again that integration of the δ -function over $\cos \theta$ effectively restricts the integration over β to between $\beta_{\min}$ and $\beta_{\max}$. The result is

$$I_{\beta} = \frac{\Omega m^*}{4\pi^2 \hbar p} \ln \left[\frac{\beta_{\max}}{\beta_{\min}} \right]$$

which can be used in eq. (2.74) to find

$$\frac{1}{\tau(p)} = \frac{q^2 \omega_0 (N_0 + \frac{1}{2} \mp \frac{1}{2}) (\frac{\kappa_0}{\kappa_{\infty}} - 1)}{4\pi \kappa_0 \epsilon_0 \hbar \sqrt{2E(p)}/m^*} \ln(\beta_{\max}/\beta_{\min}) \quad (2.76)$$

and all that remains is to specify $\beta_{\max}$ and $\beta_{\min}$.

To find the maximum and minimum phonon wave vectors, we set the argument of the δ -function in eq. (2.75) to zero and find a quadratic equation,

$$\beta^2 \pm \frac{(2p \cos \theta)}{\hbar} \beta \mp \frac{2p \omega_0}{\hbar v} = 0 \quad (2.77)$$

whose solutions give those values of β which satisfy energy and momentum conservation for a given scattering angle θ . Solving for $\beta_{\max}$ ($\cos \theta = -1$) and $\beta_{\min}$ ($\cos \theta = +1$), we find

$$\beta_{\max} = \frac{p}{\hbar} \left[1 + \sqrt{1 \pm \frac{\hbar \omega_0}{E(p)}} \right] \quad (2.78)$$

and

$$\beta_{\min} = \frac{p}{\hbar} \left[\mp 1 \pm \sqrt{1 \pm \frac{\hbar \omega_0}{E(p)}} \right]$$

which are very similar to the results of Section 2.3. With these results, the POP scattering rate becomes

$$\frac{1}{\tau(p)} = \frac{q^2 \omega_0 (\frac{\kappa_0}{\kappa_{\infty}} - 1)}{2\pi \kappa_0 \epsilon_0 \hbar \sqrt{2E(p)}/m^*} \left[N_0 \sinh^{-1} \left[\frac{E(p)}{\hbar \omega_0} \right]^{\frac{1}{2}} + (N_0 + 1) \sinh^{-1} \left[\frac{E(p)}{\hbar \omega_0} - 1 \right]^{\frac{1}{2}} \right] \quad (2.79)$$

where the first term represents POP absorption, and the second POP emission. It is understood that the second term applies only when $E(p) > \hbar \omega_0$, so that emission can occur. In going from eq. (2.76) to eq. (2.79), we made use of the identity

$$\sinh^{-1}(x) = \ln \left[x + \sqrt{1 + x^2} \right]$$

Figure 2.11 is a plot of the POP scattering rate versus carrier energy for electrons in GaAs at room temperature. The onset of phonon emission at $E(p) \approx \hbar \omega_0 \approx 35$ meV is readily apparent. Notice that in contrast to ADP and ODP scattering, the POP scattering rate is roughly constant at high energies.

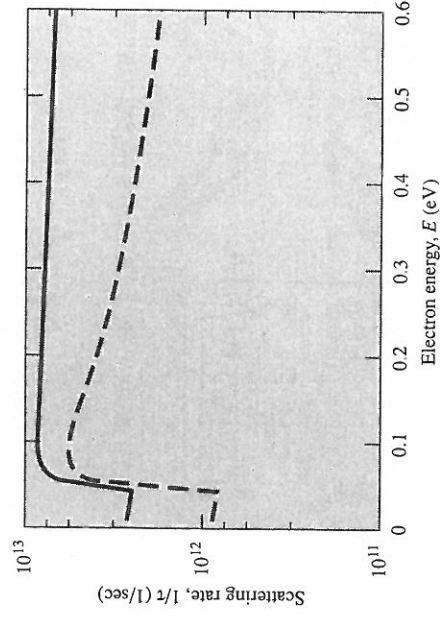


Fig. 2.11 Polar optical phonon scattering rates versus energy for electrons in GaAs at room temperature. Parabolic bands are assumed. The solid line is $1/\tau(p)$, and the dashed line is $1/\tau_m(p)$.

2.6.1 POP Energy Relaxation Time

The energy relaxation rate due to POP scattering is found by weighting each transition by the fractional change in energy. For high-energy electrons, POP emission dominates, and from eqs. (2.5) and (2.79) we find

$$\begin{aligned}\tau_E(p) &= \left[\frac{E(p)}{\hbar \omega_0} \right] \tau(p) \\ &= \frac{4\pi\kappa_0\epsilon_0 E(p) \sqrt{2E(p)/m^*}}{q^2 \omega_0^2 (\frac{\kappa_0}{\kappa_\infty} - 1)} (N_0 + 1) \sinh^{-1} \left[\frac{E(p)}{\hbar \omega_0} - 1 \right] \quad (2.80)\end{aligned}$$

2.6.2 POP Momentum Relaxation Time

The momentum relaxation time is found from an expression similar to eq. (2.4). Each transition is weighted by

$$(1 - \frac{p'}{p} \cos \alpha) = \mp \frac{\hbar \beta}{p} \cos \theta \quad (2.81)$$

where the $\hat{z}$ axis is oriented along $\mathbf{p}$ so that $\alpha = \theta$. From eqs. (2.81) and (2.4) we find

$$\frac{1}{\tau_m(p)} = \sum_{\mathbf{p}'} \left(\mp \frac{\hbar \beta}{p} \cos \theta \right) S(\mathbf{p}, \mathbf{p}') \quad (2.82)$$

which works out much like the sum in eq. (2.74), with the result

$$\begin{aligned}\frac{1}{\tau_m(p)} &= \frac{q^2 \omega_0 (\frac{\kappa_0}{\kappa_\infty} - 1)}{4\pi\kappa_0\epsilon_0 \hbar \sqrt{2[E(p)/m^*]}} \left[N_0 \sqrt{1 + \frac{\hbar \omega_0}{E(p)}} + (N_0 + 1) \sqrt{1 - \frac{\hbar \omega_0}{E(p)}} \right] \\ &\quad - \frac{\hbar \omega_0 N_0}{E(p)} \sinh^{-1} \left[\frac{E(p)}{\hbar \omega_0} \right]^{1/2} + \frac{\hbar \omega_0 (N_0 + 1)}{E(p)} \sinh^{-1} \left[\frac{E(p)}{\hbar \omega_0} - 1 \right]^{-1/2} \quad (2.83)\end{aligned}$$

The momentum relaxation rate versus energy is also plotted in Fig. 2.11. Notice that τ_m exceeds τ , which is a consequence of the fact that POP scattering favors small angle scattering events which have little effect on momentum relaxation.

2.7 INTERVALLEY SCATTERING

As displayed in Fig. 2.12, the constant energy surfaces for electrons in Si and GaAs consist of several valleys. For Si, the valleys are energetically equivalent and lie along [100] directions near the zone boundary. Two types of intervalley scattering are possible in Si. The "g-type" processes move a carrier from a given valley to one on the opposite side of the same axis (e.g., from a valley along $\langle 100 \rangle$ to one along $\langle 100 \rangle$). The "f-type" processes move a carrier to one of the remaining valleys. Both g- and f-type scattering produce very large changes in momentum, so they require phonons with wave vectors near the zone boundary. Such phonons are termed *intervalley phonons* and may be either acoustic or optical phonons. Note from Fig. 1.13a that near the zone boundary the energies of both acoustic and optical phonons are comparable and are somewhat less than the longitudinal optical phonon energy, $\hbar \omega_0$. The specific phonons involved in g- and f-type scattering are listed in Table 2.1 at the end of this chapter.

Intervalley scattering in GaAs is somewhat different because the valleys shown in Fig. 2.12b are not energetically equivalent. The central, Γ , valley lies about 0.3 eV below the ellipsoidal, L, valleys located along $\langle 111 \rangle$ directions. An illustration in energy-momentum space of a Γ -to-L transition is shown in Fig. 2.13. Because carriers must acquire ≈ 0.3 eV of energy, intervalley scattering is rare in GaAs unless a high electric field is present to accelerate the carrier to energies high within the Γ valley. For electrons in GaAs, equivalent, L-L, intervalley scattering also occurs in addition to the nonequivalent, Γ -L, scattering.

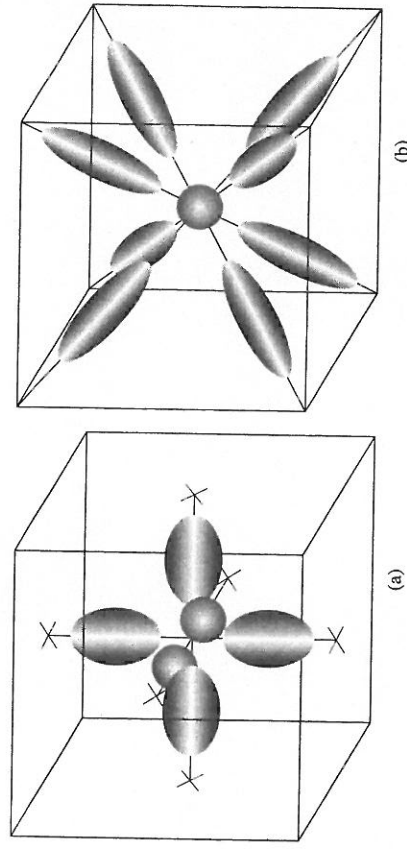


Fig. 2.12 Constant energy surfaces for (a) silicon and (b) gallium arsenide.

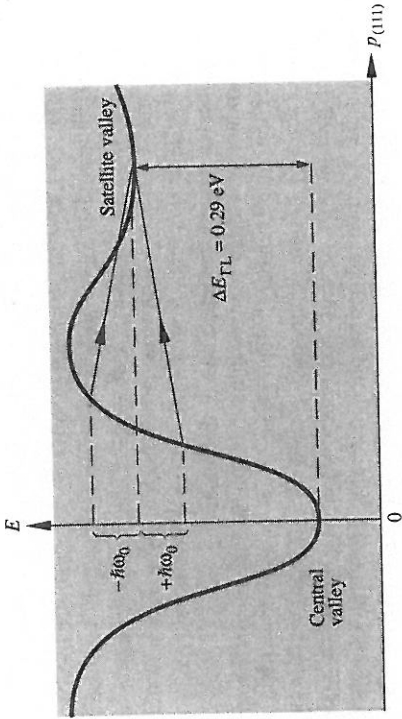


Fig. 2.13 Diagram illustrating nonequivalent intervalley scattering by phonon absorption and emission in GaAs. From Seeger, *Semiconductor Physics*, 3rd Ed., Springer-Verlag, New York, 1985.

The mathematical treatment of intervalley scattering is done in a very simple, phenomenological way. We postulate an interaction potential,

$$U_{INT} = D_{if} u(x, t) \quad (2.84)$$

where D_{if} , the intervalley deformation potential, characterizes the strength of the scattering from the initial valley "i" to the final valley "f." This interaction potential is like eq. (2.46) for optical phonon scattering, but if the intervalley phonons are acoustic phonons, we really should use an expression like eq. (2.45). Such a choice would simply change our final result by a constant, which could be absorbed in the definition of D_{ij} .

When we evaluated the scattering rates for optical phonon scattering, we assumed that the phonon energy was constant. For intervalley scattering, the phonon momentum is large and nearly constant, so $\hbar\omega_{if}$, the intervalley phonon energy, can be assumed to be constant. As a consequence, the intervalley scattering rate works out just like that for nonpolar optical phonon scattering. By analogy with eq. (2.68) we find

$$S(\mathbf{p}, \mathbf{p}') = \frac{\pi D_{if}^2 Z_f}{\rho \omega_{if} \Omega} (N_i + \frac{1}{2} \mp \frac{1}{2}) \delta(E' - E + \Delta E_{fi} \mp \hbar\omega_{if}) \quad (2.85)$$

where Z_f is the number of final valleys available for scattering. The term ΔE_{fi} is the difference between the bottom of the conduction bands in the final and initial valleys ($\Delta E_{fi} = 0$ for equivalent intervalley scattering in Si and GaAs, and $\Delta E_{fi} \approx 0.3$ eV for Γ -L, nonequivalent scattering in GaAs). N_i is the number of intervalley phonons as given by the Bose-Einstein factor.

Because intervalley scattering is isotropic, $\tau(p) = \tau_m(p)$. By analogy with eq. (2.69) we find

$$\frac{1}{\tau(p)} = \frac{1}{\tau_m(p)} = \frac{\pi D_{if}^2 Z_f}{2\rho\omega_{if}} (N_i + \frac{1}{2} \mp \frac{1}{2}) g_C(E \pm \hbar\omega_{if} - \Delta E_{fi}) \quad (2.86)$$

where the density of states is that of the final valley. In Fig. 2.14 we plot the intervalley scattering rates for Si and GaAs. For silicon, equivalent intervalley scattering is important near room temperature. For electrons in GaAs, nonequivalent intervalley scattering is important only at high fields.

2.8 CARRIER-CARRIER AND PLASMON SCATTERING

When the carrier density is high, collisions between carriers are an important scattering mechanism. Two types of processes must be distinguished — a binary process in which one carrier collides with another, and a collective process in which a carrier

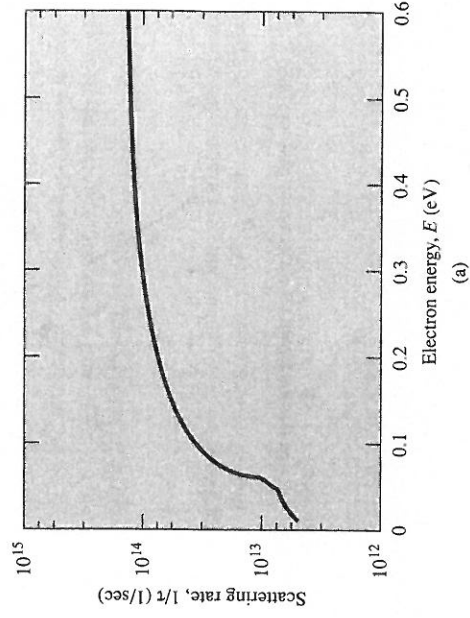


Fig. 2.14 (a) Equivalent intervalley scattering rate in silicon due to f -type and g -type scattering processes. The intervalley deformation potentials and phonon energies are listed in Table 2.1. (b) Nonequivalent, Γ -L, scattering rate in GaAs at room temperature.

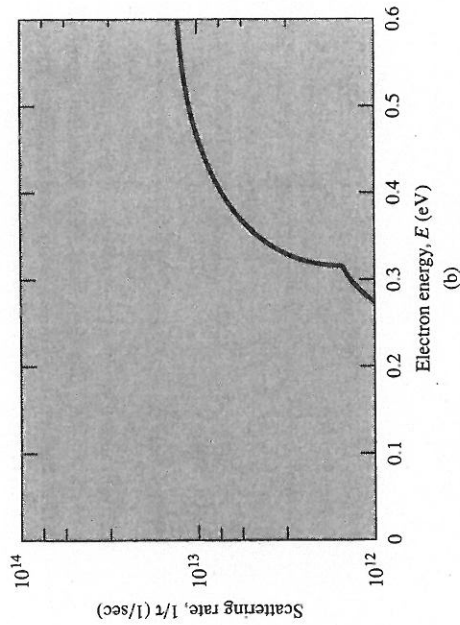
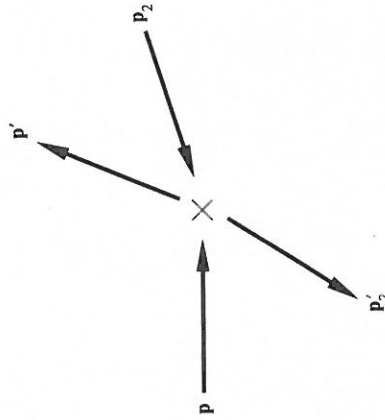


Fig. 2.14 (continued)

Fig. 2.15 Binary collision between a carrier with momentum $\mathbf{p}$ and one with momentum $\mathbf{p}_2$.

interacts with the plasma comprised by the carriers. For the binary process, which is depicted in Fig. 2.15, momentum and energy conservation dictate that

$$\mathbf{p} + \mathbf{p}_2 = \mathbf{p}' + \mathbf{p}'_2 \quad (2.87a)$$

and

$$E(\mathbf{p}) + E(\mathbf{p}_2) = E(\mathbf{p}') + E(\mathbf{p}'_2) \quad (2.87b)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the carrier before and after the collision, and $\mathbf{p}_2$ and $\mathbf{p}'_2$ refer to the carrier it collides with. Although the total momentum and energy of the carrier ensemble cannot change by carrier-carrier scattering, the distribution of momenta can be affected. By altering the distribution, carrier-carrier scattering affects the average relaxation times and, therefore, the values of observables such as the average carrier velocity and energy.

To write the collision term for carrier-carrier scattering, we define a *pair transition rate*, $S(\mathbf{p}, \mathbf{p}_2; \mathbf{p}', \mathbf{p}'_2)$, which is the probability per unit time that carriers at $\mathbf{p}$ and $\mathbf{p}_2$ collide and scatter to $\mathbf{p}'$ and $\mathbf{p}'_2$. When viewed in the center-of-mass reference frame, the binary carrier-carrier collision looks just like an ionized impurity scattering event. By analogy with eq. (2.21a) for ionized impurity scattering, we write the pair transition rate as

$$S(\mathbf{p}, \mathbf{p}_2; \mathbf{p}', \mathbf{p}'_2) = \frac{2\pi q^4 / \hbar \kappa_S^2 \epsilon_0^2 \Omega}{4(p/\hbar)^2 \sin^2(\alpha/2) + 1/L_D^2} \times \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}' - \mathbf{p}'_2) \times \delta[E(\mathbf{p}) + E(\mathbf{p}_2) - E(\mathbf{p}') - E(\mathbf{p}'_2)] \quad (2.88)$$

(For details of this derivation, consult Ridley, Ref. [2.1].)

To evaluate the scattering rate due to binary carrier-carrier scattering, we weight the pair transition rate by the probability that a target carrier is present and by the probability that the final states at $\mathbf{p}'$ and $\mathbf{p}'_2$ are empty. The result is summed over the final states, $\mathbf{p}'$ and over the target states, $\mathbf{p}_2$, to obtain

$$\frac{1}{\tau(\mathbf{p})} = \sum_{\mathbf{p}_2} \sum_{\mathbf{p}'} S(\mathbf{p}, \mathbf{p}_2; \mathbf{p}', \mathbf{p}'_2) f(\mathbf{p}_2) [1 - f(\mathbf{p}')] [1 - f(\mathbf{p}'_2)] \quad (2.89)$$

[A separate sum over $\mathbf{p}'_2$ is not performed because it is uniquely determined from $\mathbf{p}$, $\mathbf{p}_2$, and $\mathbf{p}'$ according to eq. (2.87a).] In eq. (2.89), $f(\mathbf{p})$ is the probability that the state at $\mathbf{p}$ is occupied, and is known as the distribution function. To find $f(\mathbf{p})$, a complicated integrodifferential equation known as the Boltzmann Transport Equation (BTE) must be solved. The requirement that we know the distribution function makes carrier-carrier scattering extremely difficult to treat. The BTE, and techniques to solve it for the distribution function, are the subjects of Chapter 3.

In addition to binary collisions, carriers can also interact with oscillations in the carrier density. Such fluctuations are accompanied by electric fields which oppose the fluctuation and produce oscillations at the *plasma frequency*,

$$\omega_p = \left[\frac{q^2 n}{\kappa_s \epsilon_0 m^*} \right]^{1/2} \quad (2.90)$$

which can be sustained if

$$\omega_p \tau \geq 1 \quad (2.91)$$

(to ensure that collisions don't damp out the oscillations). Because τ is on the order of one picosecond, eq. (2.91) implies that plasma oscillations can be sustained if the carrier density exceeds about 10^{17} cm^{-3} . Carriers which scatter from plasma oscillations interact with the ensemble of carriers rather than with a single carrier as in the binary process.

The charge density oscillation of the plasma can be written as

$$\rho_\beta = A_\beta e^{i(\pm\beta x - \omega_\beta t)} \quad (2.92)$$

and the electric field it produces as

$$\epsilon_\beta = \int \frac{\rho_\beta dx}{\kappa_s \epsilon_0} = \frac{\rho_\beta}{i\beta \kappa_s \epsilon_0} \quad (2.93)$$

The interaction potential is obtained by integrating again to find

$$U_{PL} = -q \int \epsilon_\beta dx = \frac{q \rho_\beta}{\kappa_s \epsilon_0 \beta^2} \quad (2.94)$$

from which the matrix element for carrier-plasma scattering is obtained as

$$H_{p',p} = \frac{q A_\beta}{\kappa_s \epsilon_0 \beta^2} \delta(\mathbf{p}' - \mathbf{p} \mp \hbar \boldsymbol{\beta}) \quad (2.95)$$

So far we have treated the plasma oscillation classically, but its energy should be quantized in units of $\hbar \omega_p$. By equating the classical, electrostatic energy to its quantum mechanical counterpart, we find

$$A_\beta^2 \rightarrow \frac{\hbar \omega_p \kappa_s \epsilon_0 \beta^2}{2\Omega} (N_p + 1/2 \mp 1/2) \quad (2.96)$$

where N_p represents the number of *plasmons* (quantized plasma oscillations), as given by the Bose-Einstein factor. When eq. (2.96) is inserted in eq. (2.95) and used in the golden rule, the transition rate is found to be

$$S(\mathbf{p}, \mathbf{p}') = \frac{\pi q^2 \omega_p}{\kappa_s \epsilon_0 \beta^2 \Omega} (N_p + 1/2 \mp 1/2) \delta(\mathbf{p}' - \mathbf{p} \mp \hbar \boldsymbol{\beta}) \delta(E' - E \mp \hbar \omega_p) \quad (2.97)$$

Since the transition rate for plasmon scattering is so similar to eq. (2.73) for POP scattering, the results of Section 2.6 may be used directly. By analogy with eq. (2.76), we find the scattering rate for plasmon scattering as

$$\frac{1}{\tau(p)} = \frac{q^2 \omega_p (N_p + 1/2 \mp 1/2)}{4\pi \kappa_s \epsilon_0 \hbar \sqrt{2E(p)}/m^*} \frac{\ln(\beta_{\max}/\beta_{\min})}{m^*} \quad (2.98)$$

where $\beta_{\max}$ and $\beta_{\min}$ are determined by energy and momentum conservation as in eq. (2.78), but some care is required when specifying $\beta_{\max}$. A large $\beta_{\max}$ refers to short wavelength oscillations, but about one Debye length is required to screen out the charge of a carrier. When $\beta_{\max}$ exceeds about $1/L_D$, the scattering should be treated as a binary collision. Since eq. (2.98) does not apply to binary collisions, $\beta_{\max}$ is replaced by β_c , which is equal to $\beta_{\max}$ from eq. (2.78) or $1/L_D$, whichever is smaller. The scattering rate for plasmon scattering becomes

$$\frac{1}{\tau(p)} = \frac{q^2 \omega_p (N_p + 1/2 \mp 1/2)}{4\pi \kappa_s \epsilon_0 \hbar \sqrt{2E(p)}/m^*} \ln \left[\frac{\hbar \beta_c / 2p}{\mp 1 \pm \sqrt{1 \pm \hbar \omega_p / E(p)}} \right] \quad (2.99)$$

In Fig. 2.16 we plot the scattering rate versus energy for electron-plasmon scattering in GaAs at room temperature. For high carrier densities, plasmon scattering is an important component of the total scattering rate, but when the electron density exceeds about 10^{18} cm^{-3} , the computations become more involved because the plasma oscillations couple with the LO phonons (Ref. [2.9]), and the scattering rate from these *coupled modes* must be evaluated.

2.9 PHONON SCATTERING OF CONFINED CARRIERS

In previous sections, we've examined the physics of scattering for three-dimensional or bulk electrons. For two-dimensional carriers confined in quantum wells, however, the scattering rate is much different. Figure 2.17 illustrates electron scattering in a GaAs quantum well. Although the electrons may lie in the central, Γ valley, they can occupy different subbands within the well. We need to consider both *intra-subband* and *inter-subband* scattering of electrons in quantum wells. The calculation of these scattering rates proceeds much like it does for 3-D electrons, but the proper wave function for confined carriers must be used. We should also consider the possibility of 2-D phonons

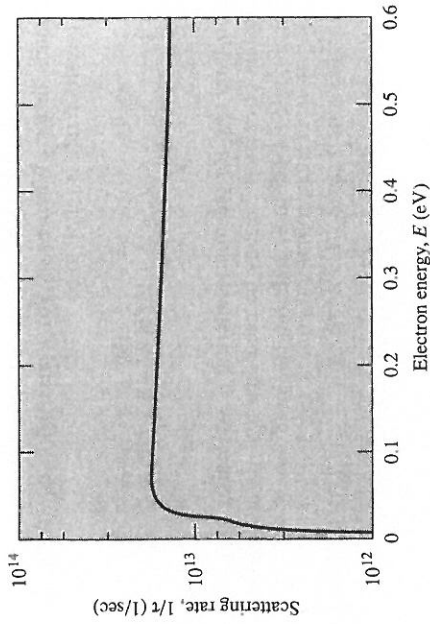


Fig. 2.16 Electron-plasmon scattering rate versus energy for electrons in GaAs at room temperature. The electron density is $n_0 = 1.0 \times 10^{17} \text{ cm}^{-3}$.

as well as 2-D electrons, but for many quantum wells, the elastic constants of the well are similar to those of the surrounding media, so 3-D phonons can be assumed. In this section, we'll discuss ADP scattering of 2-D electrons to illustrate how such calculations proceed and to establish the important features of phonon scattering of confined carriers.

We begin by assuming an infinitely deep quantum well, so that the wave function of the confined electrons is

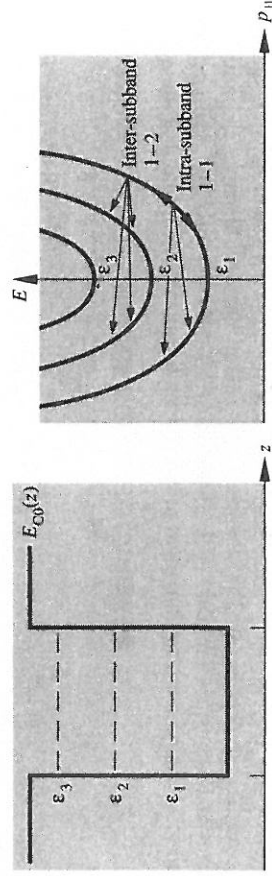


Fig. 2.17 Illustration of intra-subband and inter-subband scattering of electrons in a quantum well. Scattering by acoustic phonons is assumed.

$$\psi(x, y, z) = \sqrt{\frac{2}{AW_z}} e^{ip_{\parallel} \cdot \mathbf{p}/\hbar} \sin(k_z z) \quad (2.100)$$

where $\mathbf{p}_{\parallel}$ is momentum in the x - y plane, and $\mathbf{p}$ is the position vector in the x - y plane. The boundary conditions restrict k_z to integer multiples of π/W_z . Using this wave function and the interaction potential for ADP scattering [eq. (2.45)], we find the matrix element as

$$H_{p'p} = (\pm i\beta D_A) A_{\beta} \delta(\mathbf{p}'_{\parallel} - \mathbf{p}_{\parallel} \mp \hbar\beta_{\parallel}) G(\beta_z) \quad (2.101)$$

where

$$G(\beta_z) \equiv \frac{2}{W_z} \int_0^{W_z} e^{\pm i\beta_z z} \sin(k_z z) \sin(k'_z z) dz \quad (2.102)$$

After using prescription [eq. (2.59)] for A_{β}^2 and assuming elastic scattering and equipartition, we find the transition rate as

$$S(\mathbf{p}, \mathbf{p}') = \left[\frac{2\pi D_A^2 k_B T_L}{\hbar c_l A} \right] \delta(\mathbf{p}'_{\parallel} - \mathbf{p}_{\parallel} \mp \hbar\beta_{\parallel}) \delta(E' - E) |G(\beta_z)|^2 \quad (2.103)$$

Equation (2.103) should be compared with eq. (2.61), the corresponding result for ADP scattering of 3-D electrons. The two expressions are identical except for the replacement,

$$\frac{1}{\Omega} \delta(\mathbf{p}' - \mathbf{p} \mp \hbar\beta) \rightarrow \frac{1}{A} \delta(\mathbf{p}'_{\parallel} - \mathbf{p}_{\parallel} \mp \hbar\beta_{\parallel}) |G(\beta_z)|^2 \quad (2.104)$$

For three-dimensional electrons momentum is conserved, but for two-dimensional electrons, it is conserved only in the x - y plane. The result should have been anticipated because the uncertainty principle states that

$$\Delta p_z \geq \hbar/W_z$$

Because the uncertainty in the particle's location in the $\hat{z}$ -direction has been reduced by confining it in a well, there must be a corresponding increase in the uncertainty in the particle's $\hat{z}$ -directed momentum.

To find the scattering rate for ADP scattering of 2-D electrons, we evaluate

$$\frac{1}{\tau(\mathbf{p})} = \frac{1}{\tau_m(\mathbf{p})} = \sum_{\mathbf{p}'} S(\mathbf{p}, \mathbf{p}') = \sum_{\mathbf{p}', \mathbf{p}''} S(\mathbf{p}, \mathbf{p}') \quad (2.105)$$

which is difficult to evaluate because of the presence of $|G(\beta_z)|^2$. For very wide wells, $|G(\beta_z)|^2$ reduces to a δ -function expressing momentum conservation in the $\hat{z}$ -direction, but for narrow wells it has to be evaluated numerically. If we assume momentum conservation and replace $|G(\beta_z)|^2$ by a δ -function, the sum is readily evaluated. This *momentum conservation approximation* (MCA) is accurate as long as the well is not too narrow (it should be ≥ 100 Å for electrons in a GaAs well; Ref. [2.11]). With the MCA we can obtain closed-form expressions for the scattering rates of 2-D electrons and gain considerable insight into the nature of electron scattering in quantum wells.

Converting $|G(\beta_z)|^2$ to a δ -function is a bit tricky, but the result can be understood with the aid of Fig. 2.18 (consult Ridley, Ref. [2.11], for details of the derivation). For electrons in a quantum well, $k_z = n\pi/W_z$, where n labels the subband. The electron states are standing waves, which are a superposition of two traveling waves — one with wave vector $+k_z$ and a second with $-k_z$. Figure 2.18a lists the four possible transitions; two of them leave the electron's direction of propagation unchanged and two of them reflect the carrier. The transitions illustrated in Fig. 2.18a are inter-subband transitions because the magnitude of k_z changes. For this case we find (Ref. [2.11])

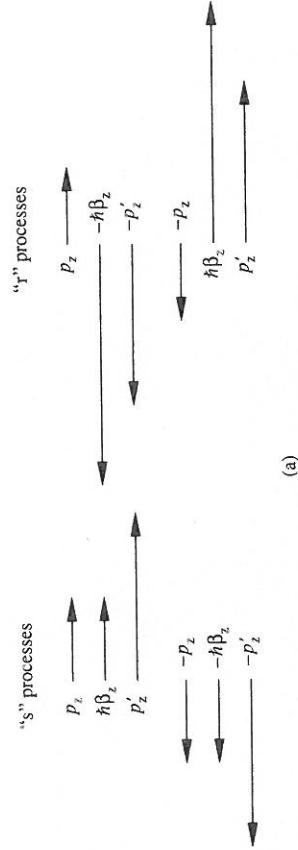


Fig. 2.18 Illustrations of possible (a) inter-subband and (b) intra-subband scattering transitions for electrons in a quantum well. Processes labeled “s” conserve momentum without changing the direction of propagation, and those labeled “r” produce a reflection of the wave vector.

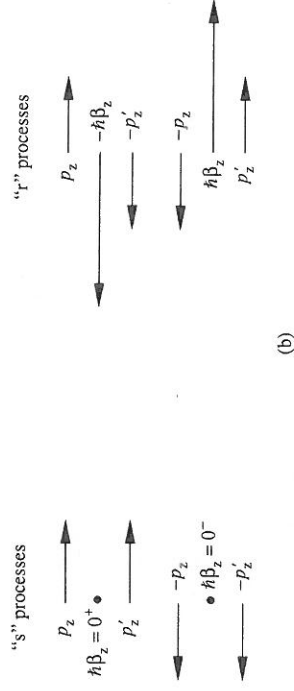


Fig. 2.18 (continued)

$$|G(\beta_z)|^2 = \frac{1}{4} \left[\delta(p'_z - p_z \mp \hbar\beta_z) + \delta(-p'_z + p_z \pm \hbar\beta_z) \right]$$

$$+ \delta(p'_z - p_z \pm \hbar\beta_z) + \delta(-p'_z + p_z \mp \hbar\beta_z) \quad (2.106)$$

and so

$$\sum_{\beta_z} |G(\beta_z)|^2 = 1 \quad (2.107)$$

Figure 2.18b illustrates transitions for intra-subband scattering. In this case, k_z does not change in magnitude. Note that there are two possibilities for transitions without reflections. These two possibilities constructively interfere with one another, so the amplitude for the process is doubled and the probability of the transition is multiplied by four. The result is (Ref. [2.11])

$$|G(\beta_z)|^2 = \frac{1}{4} \left[4\delta(p'_z - p_z) + \delta(p'_z - p_z \pm \hbar\beta_z) + \delta(-p'_z + p_z \mp \hbar\beta_z) \right]$$

and so

$$\sum_{\beta_z} |G(\beta_z)|^2 = \frac{3}{2} \quad (2.107)$$

We can combine these two results and find

$$\sum_{\mathbf{p}_z} |G(\beta_z)|^2 = \frac{(2 + \delta_{i,f})}{2} \quad (2.108)$$

where “i” and “f” label the initial and final subbands and $\delta_{i,f}$ is the Kronecker delta, which is zero when $i \neq f$ and one when $i = f$.

Having approximated $|G(\beta_z)|^2$ by δ -functions, we are ready to evaluate the sum [eq. (2.105)]. By summing first over β_z (rather than $\mathbf{p}'_z$), we find

$$\frac{1}{\tau_m(\mathbf{p})} = \left[\frac{2\pi D_A^2 k_B T_L}{\hbar c_l} \right] \frac{(2 + \delta_{i,f})}{2} \frac{1}{A} \sum_{\mathbf{p}'_{\parallel}} \delta(\mathbf{p}'_{\parallel} - \mathbf{p}_{\parallel} \mp \hbar \beta_{\parallel}) \delta(E' - E) \quad (2.109)$$

The sum in eq. (2.109) should be recognized as the two-dimensional density of states, so the final result is

$$\frac{1}{\tau_{i,f}} = \left[\frac{2\pi D_A^2 k_B T_L}{\hbar c_l} \right] \frac{(2 + \delta_{i,f})}{2} \frac{g_{2Df}(E)}{2} \quad (2.110a)$$

Equation (2.110a) describes ADP scattering from subband “i” to subband “f”; g_{2Df} is the two-dimensional density of states for subband “f.” For intra-subband scattering, eq. (2.110a) reduces to

$$\frac{1}{\tau_{i,i}} = \left[\frac{3\pi D_A^2 k_B T_L}{\hbar c_l} \right] \frac{g_{2Di}(E)}{2} \quad (2.110b)$$

and for inter-subband scattering, to

$$\frac{1}{\tau_{i,f}} = \left[\frac{2\pi D_A^2 k_B T_L}{\hbar c_l} \right] \frac{g_{2Df}(E)}{2} \quad (2.110c)$$

In Chapter 1, we showed that the two-dimensional density of states per subband is constant with energy. Adding the density of states for each subband, we get the result plotted in Fig. 2.19a. The 2-D ADP scattering rate is proportional to the 2-D density of states, so for intra-subband ADP scattering, we get the result plotted in Fig. 2.19b. Inter-subband transitions can also be induced by ADP scattering; the acoustic phonons carry little energy, so an electric field must accelerate carriers to energies exceeding the bottom of the next subband before such scattering can occur. Figure 2.19c illustrates inter-subband scattering of electrons by acoustic phonons.

In this section, we’ve discussed one example calculation of scattering rates for carriers confined in a quantum well. For such carriers, the matrix element needs to be

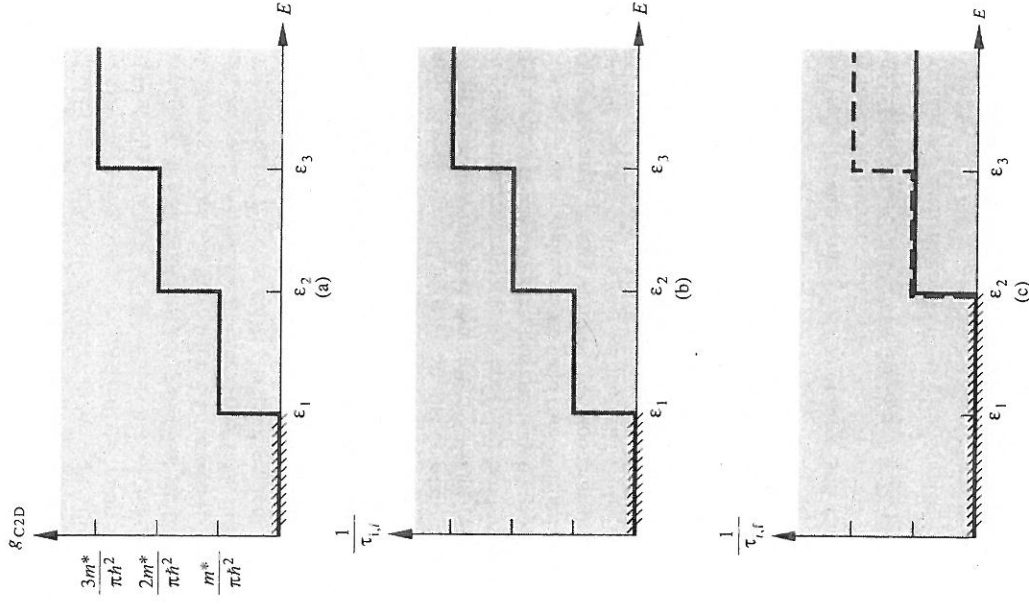


Fig. 2.19 (a) The two-dimensional density of states versus energy for electrons in a quantum well. The subband energies are labeled ϵ_1 , ϵ_2 , and ϵ_3 . (b) Acoustic deformation potential scattering rate versus energy for 2-D electrons. Intra-subband scattering is assumed. (c) Inter-subband scattering by acoustic phonons. The solid line is for scattering from subband 1 to subband 2, and the dashed line is the total scattering rate for transitions from subband 1 to subband 2 and from subband 1 to subband 3.

evaluated with the proper wave function for confined carriers — not with the plane wave functions used for bulk electrons. We found that momentum conservation applies in the plane of the quantum well but that it is “fuzzy” in the $\hat{z}$ -direction. To evaluate the sum required to find the scattering rate, we invoked the momentum conservation approximation, which is accurate as long as the well is not too narrow. Both 2-D and 3-D scattering rates generally vary as the density of states, but the staircase variation of g_{2D} with energy leads to very abrupt changes in the 2-D scattering rate versus energy. Similar procedures can be used to evaluate the scattering rates due to other mechanisms. For example, the transition rates for ODP and POP scattering are easily found from the prescription [eq. (2.104)] which we derived for ADP scattering.

2.10 SCATTERING RATES FOR NONPARABOLIC ENERGY BANDS

When working out expressions for scattering rates, we have been assuming spherical, parabolic energy bands. For silicon and germanium, the constant energy surfaces are ellipsoids, but appropriate averages of band structure dependent parameters can often be used to reduce the problem to an “equivalent,” spherical band problem (Refs. [2.5] and [2.7]). Under high applied fields, however, carriers may be accelerated to energies far above the band minima. At such energies, the bands are definitely not parabolic. To evaluate scattering rates for high energy carriers, the nonparabolicity of the energy bands must be considered. In this section, we outline briefly how scattering rates are evaluated when the energy bands are spherical but nonparabolic.

Any spherical energy band can be described by

$$\frac{p^2}{2m^*} = \gamma[E(p)] \quad (2.111a)$$

where m^* is evaluated from the curvature of $E(p)$ at $p = 0$ and γ is some function of energy. For spherical, parabolic bands,

$$\gamma(E) = E(p) \quad (2.111b)$$

but if nonparabolicity is described by eq. (1.43), then

$$\gamma(E) = E(p)[1 + \alpha E(p)] \quad (2.111c)$$

To compute scattering rates, we first need to find $S(\mathbf{p}, \mathbf{p}')$ from Fermi's golden rule. When the bands are spherical and parabolic, the overlap integral $I(\mathbf{p}, \mathbf{p}')$ is unity, but for nonparabolic bands it can be substantially less than one.

The occurrence of nonunity overlap integrals is only one consequence of nonparabolicity. The various sums also work out differently. Consider the sum

$$\frac{1}{\Omega} \sum_{\mathbf{p}', \uparrow} \delta[E(\mathbf{p}') - E(\mathbf{p})] = \frac{1}{2\pi^2 \hbar^3} \int_0^\infty \delta[E(\mathbf{p}') - E(\mathbf{p})] p'^2 dp' \quad (2.112)$$

which we worked out in Section 2.1 and found the result to be one-half of the density of states. For nonparabolic bands,

$$p'^2 = 2m^* \gamma(E')$$

and so

$$p'^2 dp' = \sqrt{2\gamma(E')} (m^*)^{3/2} \frac{d\gamma'}{dE'} dE' \quad (2.113)$$

After substituting eq. (2.113) into eq. (2.112), we find

$$\frac{1}{\Omega} \sum_{\mathbf{p}', \uparrow} \delta[E(\mathbf{p}') - E(\mathbf{p})] = \frac{(2m^*)^{3/2}}{4\pi^2 \hbar^3} \sqrt{\gamma(E')} \frac{d\gamma'}{dE'} \bigg|_{E'=E} \quad (2.114)$$

which is the density of states (for one of the two spins) for a nonparabolic band.

With the technique outlined above, band nonparabolicity can be accounted for in the various scattering rates. It is essential to do so when high-field transport is analyzed. For example, when nonparabolicity is included, we find that the POP scattering rate is nearly constant with energy (for energies considerably above the optical phonon energy) in contrast to the decreasing behavior displayed in Fig. 2.11, which was based on parabolic energy bands.

2.11 ELECTRON SCATTERING IN INTRINSIC SI AND GaAs

The expressions we've developed in this chapter describe electron scattering in common semiconductors. We haven't treated hole scattering because it is complicated by the degenerate heavy and light hole bands with their warped constant energy surfaces (Ref. [2.6]). For energetic carriers, overlap integrals need to be considered and the nonparabolicity accounted for, as discussed in Section 2.10. For very energetic carriers, even this is not adequate, and a detailed, numerical description of $E(\mathbf{p})$ is required.

Common scattering mechanisms can be classified as shown in Fig. 2.20. The total scattering rate is the sum of the rates for each of the individual processes,

$$\Gamma(\mathbf{p}) = \sum_i \frac{1}{\tau_i(\mathbf{p})} \quad (2.115)$$

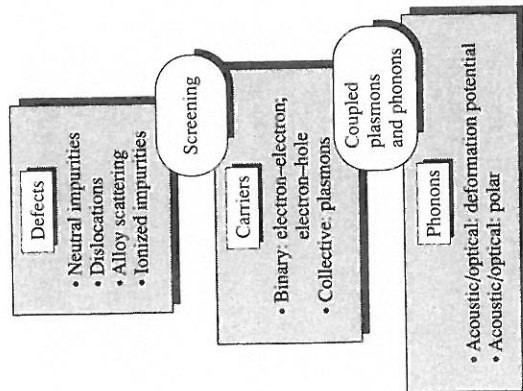


Fig. 2.20 Classification of carrier scattering mechanisms.

where the index, i , labels the various scattering mechanisms listed in Fig. 2.20. To evaluate the scattering rate versus energy, the important scattering mechanisms need to be identified for the particular semiconductor and conditions under consideration. Scattering occurs by defects, by phonons, and by other carriers. Defect scattering includes scattering by both ionized and neutral impurities and by crystal defects such as dislocations. For semiconductor alloys, variations in the alloy composition also produce scattering. Phonon scattering occurs by the deformation potential in covalent semiconductors and by both the deformation potential and polar interactions in compound semiconductors. Carrier-carrier scattering includes both binary collisions and interactions with the carrier plasma. Free carriers can also influence the other scattering processes by screening the perturbing potential. In polar semiconductors, free carrier plasma oscillations can also couple with the longitudinal optical phonons. For high-quality, intrinsic, crystalline semiconductors, defect scattering is minimal as is carrier-carrier scattering if the carrier density is low. Figure 2.21 plots electron scattering rates versus energy for intrinsic Si and GaAs. For these conditions, scattering is dominated by intra- and intervalley phonon scattering.

For pure silicon, acoustic deformation potential and equivalent intervalley scattering are the dominant mechanisms. Near room temperature, the acoustic deformation and equivalent intervalley scattering rates for thermal average electrons are comparable. Because of the several phonons involved, intervalley scattering rises faster than $\sqrt{E}$ as

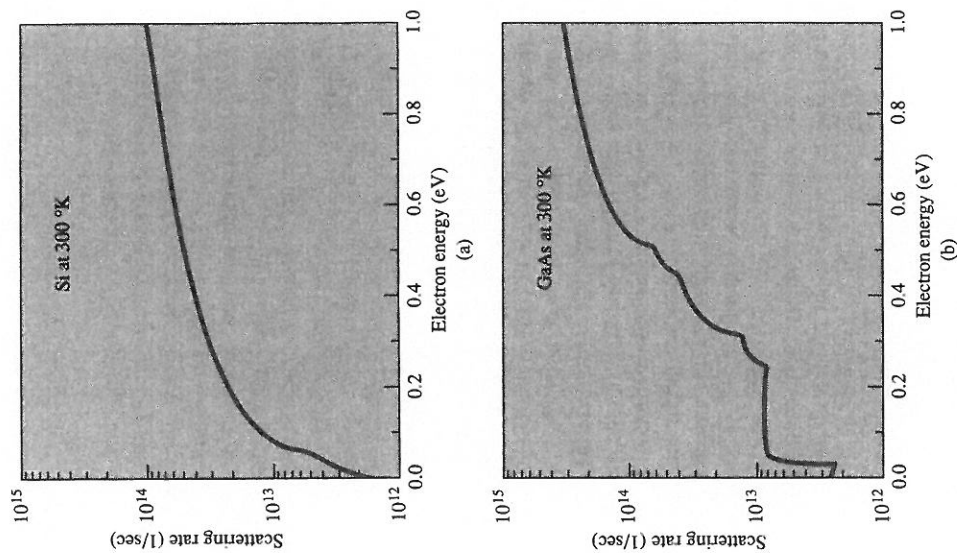


Fig. 2.21 Electron scattering rates versus energy for (a) intrinsic Si and (b) intrinsic GaAs, both at room temperature. Curve (a) was computed by assuming ellipsoidal, nonparabolic energy bands, and curve (b) was computed by assuming spherical, nonparabolic energy bands. Courtesy of Amitava Das.

the onset for various emission processes is met.¹ As a consequence, the high energy scattering rate is dominated by equivalent intervalley scattering.

The scattering rate versus energy for electrons in intrinsic GaAs is displayed in Fig. 2.21b and shows a characteristic distinctly different from that of Si. The important scattering mechanisms in intrinsic GaAs are POP intravalley and Γ -L and L-L intervalley scattering. The scattering rate displays two thresholds: the first (at $E \approx 0.03$ eV) is for POP emission and the second (at $E \approx 0.3$ eV) is for Γ -L intervalley scattering. In contrast to Si, the scattering rate remains low until the onset of intervalley scattering, when the two become comparable. We'll see in later chapters that many of the features of transport in Si and GaAs can be understood from their scattering characteristics as summarized in Fig. 2.21.

2.12 SUMMARY

The purpose of this chapter was to illustrate the procedures and a few of the tricks commonly employed to evaluate scattering rates for carriers in semiconductors. Important scattering mechanisms for common semiconductors were also identified, and the scattering rate versus energy characteristics of pure Si and GaAs were described. The method employed is based on Fermi's golden rule and proceeds in a straightforward manner once the interaction potential is identified. The scattering rate, and the momentum and energy relaxation rates are evaluated directly from the transition rate, $S(\mathbf{p}, \mathbf{p}')$. Various materials parameters needed for scattering rate calculations in silicon and gallium arsenide are listed in Table 2.1. Although our discussion centered on electron scattering in Si and GaAs, the general features of the results are typical of those observed in other covalent and polar semiconductors.

Since our motives were to introduce the basic procedure and to establish the approximate magnitudes of the various scattering rates and their functional dependence on carrier energy, a simple approach was adopted. In practice, however, overlap integrals must be treated and the energy bands cannot be described as spherical and parabolic. For high energy electrons in GaAs, the conduction band nonparabolicity is important, and for electrons in Si, the ellipsoidal nature of the conduction band minima must be included. For holes, scattering between light and heavy hole bands must be included along with the warped constant energy surfaces described in Chapter 1. Many of these refinements to the basic procedure are discussed in the chapter references.

Table 2.1 Band Parameters for Silicon and Gallium Arsenide.

Parameter	Symbol	Value in Si	Value in GaAs
Mass density (g/cm ³)	ρ	2.329	5.36
Lattice constant ($\text{\AA}$)	a_0	5.43	5.642
Low frequency dielectric constant	κ_0	11.7	12.90
High frequency dielectric constant	κ_∞	—	10.92
Piezoelectric constant (C/m ²)	e_{pz}	—	0.160
Longitudinal acoustic velocity ($\times 10^5$ cm/sec)	v_l	9.04	5.24
Transverse acoustic velocity ($\times 10^5$ cm/sec)	v_t	5.34	3.0
Longitudinal optical phonon energy (eV)	$\hbar\omega_0$	0.063	0.03536
Electron effective mass ratio (lowest valley)	m^* (m_l^*, m_t^*)	— (0.916, 0.19)	0.067 —
Electron effective mass ratio (upper valley)	m^* m_l^*, m_t^*	— 1.59, 0.12	0.222 (L) 0.58 (X) —
Nonparabolicity parameter (eV ⁻¹)	α	0.5	0.610 (T) 0.461 (L) 0.204 (X)
Electron acoustic deformation potential (eV)	D_A	9.5	7.0 (T) 9.2 (L) 9.0 (X)
Electron optical deformation potential ($\times 10^8$ eV/cm)	D_O	—	3.0 (L)
Optical phonon energy (eV)	ω_0		0.0343 (L)
Hole acoustic deformation potential (eV)	D_A	5.0	3.5
Hole optical deformation potential (eV/cm)	D_O	6.00	6.48

Table 2.1 (continued)

Parameter	Symbol	Value in Si	Value in GaAs
Intervalley parameters, g-type (X-X)	D_{ij}, E_{ij} ($\times 10^8$ eV/cm) (eV)	0.5, 0.8, 11.0,	—
Intervalley parameters, f-type (X-X)	D_{ij}, E_{ij} ($\times 10^8$ eV/cm), (eV)	0.3 2.0 2.0	—
Intervalley parameters (X-L)	D_{ij}, E_{ij} ($\times 10^8$ eV/cm), (eV)	2.0 2.0 2.0 2.0	—
Intervalley deformation potential ($\times 10^8$ eV/cm)	D_{TL}, D_{TX} D_{LL}, D_{LX} D_{XX}	—	10.0, 10.0 10.0, 5.0 7.0
Intervalley phonon energy (eV)	E_{TL}, E_{TX} E_{LL}, E_{LX} E_{XX}	—	0.0278, 0.0299 0.0290, 0.0293 0.0299
Energy separation between valleys (eV)	ΔE_{LT} ΔE_{XT}	—	0.29 0.48

Sources: Jacoboni and Reggiani, Ref. [2.5], and Hess, *Advanced Theory of Semiconductor Devices*, Prentice-Hall, Englewood Cliffs, NJ, 1988.

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PROBLEMS

- 2.1 (a) Show that the Brooks-Herring scattering rate,

$$\frac{1}{\tau(p)} = \sum_{\mathbf{p}', \mathbf{f}} \frac{S(\mathbf{p}, \mathbf{p}')}{p' \cdot \mathbf{f}}$$

$$\tau(p) = \frac{32\sqrt{2}m^* \pi \kappa_0^2 \epsilon_0^2}{N_i q^4} \frac{(1 + \gamma^2)}{\gamma^4} E^{3/2}(p)$$

is

- (b) Evaluate the scattering rate for thermal average electrons in room temperature GaAs doped at $N_D = 10^{17} \text{ cm}^{-3}$. Explain why the scattering rate is so much higher than the momentum relaxation rate plotted in Fig. 2.4.

2.2 (a) Show that the Conwell-Weisskopf scattering rate is

$$\frac{1}{\tau(p)} = N_i \pi b_{\max}^2 \frac{\sqrt{2m^* E(p)}}{m^*}$$

- (b) Provide a simple, physical explanation of the Conwell-Weisskopf expression for $1/\tau(p)$ in terms of the cross section for scattering, $\pi b_{\max}^2$.
- (c) Evaluate and plot the scattering rate for thermal average electrons in GaAs at room temperature and compare it with the momentum relaxation rate. Your plot should cover $10^{14} < N_i < 10^{18} \text{ cm}^{-3}$. Explain why $\tau(p)$ and $\tau_m(p)$ differ, and explain why $\tau_m(p)$ decreases with N_i more rapidly than does $\tau(p)$.

2.3 Equation (2.14), the Debye length, is the screening length for a nondegenerate semiconductor. Derive a more general expression for the screening length by removing the assumption that the semiconductor is nondegenerate.

2.4 Using arguments similar to those in Section 2.4, derive an expression for the interaction potential for piezoelectric scattering. Begin with

$$D = \kappa_s \epsilon_0 \nabla^2 + e_{PZ} \frac{\partial u}{\partial x}$$

where e_{PZ} is the *piezoelectric constant*. Show that the piezoelectric scattering potential is

$$U_{PZ}(x, t) = \frac{q e_{PZ}}{\kappa_s \epsilon_0} u(x, t)$$

This result is sometimes stated in terms of the *electromechanical coupling coefficient*, K^2 , where

$$\frac{K^2}{(1 - K^2)} \equiv \frac{e_{PZ}^2}{\kappa_s \epsilon_0 v_l}$$

and v_l is the longitudinal sound velocity.

2.5 Use the scattering potential for piezoelectric scattering derived in problem 2.4 to answer the following:

- (a) Assume equipartition and show that the matrix element for piezoelectric scattering is

$$|H_{p'p}|^2 = \left(\frac{q e_{PZ}}{\kappa_s \epsilon_0} \right)^2 \frac{k_B T_L}{2 c_l \beta^2 \Omega}$$

- (b) Write an expression for $S(\mathbf{p}, \mathbf{p}')$ for piezoelectric scattering.
- (c) Evaluate $1/\tau_m(p)$ assuming that piezoelectric scattering is elastic.

2.6 For alloys of compound semiconductors such as $\text{Al}_x\text{Ga}_{1-x}\text{As}$ microscopic fluctuations in the alloy composition, x , produce perturbations in the conduction and valence band edges. The matrix element for alloy scattering is

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi}{\hbar} \left(\frac{3\pi^2}{16} \right) \frac{|\Delta U|^2}{N \Omega} \delta(E' - E)$$

where N is the concentration of atoms and

$$\Delta U = x(1-x)(\chi_{\text{GaAs}} - \chi_{\text{AlAs}})$$

- (a) Explain why the alloy scattering rate vanishes at $x = 0$ and at $x = 1$.
- (b) Derive an expression for $\tau_m(p)$ for alloy scattering.

2.7 Acoustic phonon scattering was assumed to be elastic during our evaluation of the momentum relaxation rate [eq. (2.66)]. Work out an expression for $1/\tau_m(p)$ due to ADP scattering *without* assuming that the scattering is elastic, and show that the result is nearly equal to eq. (2.66) near room temperature. **Hint:** Review the treatment of POP scattering in Section 2.6.

2.8 Compute the energy relaxation rate due to ADP scattering. Assume energetic carriers so that phonon emission dominates, and assume that spontaneous emission dominates so that $N_{\omega_i} + 1 \approx 1$.

- (a) Show that

$$\tau_E = \frac{k_B T_L}{2m^* v_s^2} \tau_m$$

- (b) Evaluate the ratio for thermal average electrons at $T_L = 300^\circ \text{K}$ and at $T_L = 77^\circ \text{K}$.

2.9 Compute and compare the momentum relaxation times due to ionized impurity scattering under the following circumstances:

- (a) Find $1/\tau_m$ for electrons with the thermal average energy $E = 3k_B T_L / 2$ in GaAs doped at $N_D = 10^{18} \text{ cm}^{-3}$. Assume a lattice temperature of $T_L = 300^\circ \text{K}$.
- (b) Find $1/\tau_m$ for an electrons with $E = 0.3 \text{ eV}$ in GaAs doped at $N_D = 10^{18} \text{ cm}^{-3}$. Such electrons can be produced by a heterojunction launching ramp as displayed in Fig. 3.2.